SEMINAR 2025: GEOMETRIZATION OF COHOMOLOGY THEORIES

The aim of this seminar is to become familiar with the techniques of "geometrization" of cohomology theories, such as de Rham cohomology, prismatic cohomology, and syntomic cohomology, and to discuss applications. The basic idea goes back to Simpson [Sim96] on the de Rham space in characteristic zero. Then, in the last few years, the geometrization of p-adic cohomology theories has been actively developed by Drinfeld [Dri22, Dri24] and Bhatt–Lurie [BL22, BL∞], and it is the main focus of this seminar.

By geometrization of a cohomology theory E, we mean a natural assignment of a stack X^E to a scheme X such that the cohomology $\mathrm{H}^*(X^E, \mathcal{O})$ of the structure sheaf $\mathcal{O}$ on X^E recovers the E-cohomology $E^*(X)$ of X. We would like to highlight two advantages of this perspective:

— The coefficient system for E is obtained for free as quasi-coherent modules over X^E, e.g., D-modules over X are identified with quasi-coherent modules over the de Rham stack X^{dR}.

— Various structures of such a cohomology theory E and its coefficient system are understood via the geometry of the associated stack X^E (or rather the ring stack behind it, which is independent of X).

As applications, we cover Drinfeld's refinement of the Deligne–Illusie theorem as well as recent work by Petrov [Pet25], and hopefully more in due course.

We will mostly follow Bhatt's notes [Bha22] in this seminar, and the numbering below is as in his notes.

Talk 1 (Introduction, and preliminaries on filtered modules [§2.2.1])**.** In addition to the introduction, this talk covers some preliminaries on filtered modules. Explain that filtered modules are identified with quasi-coherent modules over $\mathbb{A}^1/\mathbb{G}_m$ (Proposition 2.2.6, see also [Mou21]). This allows us to interpret stacks over $\mathbb{A}^1/\mathbb{G}_m$ as "filtered stacks" (Remark 2.2.10).

Talk 2 (de Rham cohomology in characteristic zero [§2.2.2, §2.3])**.** The goal of this talk is to introduce the filtered de Rham stack in characteristic zero and to prove that it recovers the de Rham cohomology with the Hodge filtration (Theorem 2.3.6). The key input for the proof is the following: for a finite locally free sheaf $\mathcal{E}$ on an (affine?) $\mathbb{Q}$-scheme, there is an equivalence $\mathrm{QCoh}(B\widehat{\mathbb{V}(\mathcal{E})}) \simeq \mathrm{QCoh}(\mathbb{V}(\mathcal{E}^\vee))$ (Proposition 2.2.13), where $\widehat{\mathbb{V}(\mathcal{E})}$ is the formal completion of $\mathbb{V}(\mathcal{E})$ along the zero section, and it gives a recipe for computing the cohomology of quasi-coherent modules over $B\widehat{\mathbb{V}(\mathcal{E})}$ (Remark 2.2.14).

Introduce the ring stack $\mathbb{G}_a^{\mathrm{dR}}$; for a commutative ring R, we have $\mathbb{G}_a^{\mathrm{dR}}(R) = R_{\mathrm{red}}$ by Lemma 2.2.17. As its filtered refinement, introduce the filtered ring stack $\mathbb{G}_a^{\mathrm{dR},+}$ with the associated graded stack denoted by $\mathbb{G}_a^{\mathrm{Hodge}}$ (Construction 2.3.4); here the notion of *quasi-ideal* should be explained in due course (see [Dri21]).

Let k be a field of characteristic zero. For a k-scheme X, define the filtered de Rham stack $X^{\mathrm{dR},+}$ as a filtered k-stack (Definition 2.3.5) via "transmutation" from $\mathbb{G}_a^{\mathrm{dR},+}$ (Remark 2.3.8). Prove that, for a smooth qcqs k-scheme X, the direct image of the structure sheaf under the structure map $X^{\mathrm{dR},+} \to \mathbb{A}^1/\mathbb{G}_m$ recovers the Hodge-filtered de Rham cohomology of X (Theorem 2.3.6). Explain that quasi-coherent modules over $X^{\mathrm{dR},+}$ correspond to "filtered D-modules" over X (Remark 2.3.7, see also [GR14]).

Talk 3 (de Rham cohomology of p-adic formal schemes [§2.4, §2.5])**.** This talk explains the p-adic analogue of Talk 2. Things are mostly parallel to the characteristic zero case, but the computational input is slightly different: for a finite locally free sheaf $\mathcal{E}$ on a scheme, consider the PD-hull $\mathbb{V}(\mathcal{E})^\#$ of the zero section in $\mathbb{V}(\mathcal{E})$, and then there is an equivalence $\mathrm{QCoh}(B\mathbb{V}(\mathcal{E})^\#) \simeq \mathrm{QCoh}(\widehat{\mathbb{V}(\mathcal{E}^\vee)})$ (Proposition 2.4.5). Explain this by sketching the proof of the case $\mathcal{E} = \mathcal{O}$ as in Proposition 2.4.4.

Introduce the filtered formal ring stack $\mathbb{G}_a^{\mathrm{dR},+}$ and its variants (Definition 2.5.1), and explain the formula for their R-points for a p-nilpotent ring R by Lemma 2.4.7. Let V be a p-complete commutative ring with bounded p-power torsion. For a p-adic formal V-scheme X, define the filtered de Rham stack $X^{\mathrm{dR},+}$ as a filtered formal V-stack (Definition 2.5.3) via transmutation from $\mathbb{G}_a^{\mathrm{dR},+}$. Prove that, for a smooth qcqs p-adic formal V-scheme X, the direct image of the structure sheaf under the structure map $X^{\mathrm{dR},+} \to \mathbb{A}^1/\mathbb{G}_m$ recovers the Hodge-filtered de Rham cohomology of X (Theorem 2.5.6). Discuss the coefficient system for the filtered

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de Rham cohomology thus obtained (Remark 2.5.8 and 2.5.9). As an application of Theorem 2.5.6, prove the "crystalline miracle" (Corollary 2.5.10), and mention the crystalline Frobenius thus obtained (Remark 2.5.11).

Talk 4 (The conjugate filtration on de Rham cohomology [§2.6, §2.7.1]). The goal of this talk is to "geometrize" what lies behind the conjugate filtration of de Rham cohomology in characteristic p. The Witt vector model for the ring stack $\mathbb{G}_a^{\mathrm{dR}}$ will naturally lead to this.

Recall Witt vectors briefly, and describe $\mathbb{G}_a^{\#}$ in terms of Witt vectors as in Lemma 2.6.1. Using this description, give the Witt vector model for $\mathbb{G}_a^{\mathrm{dR}}$ (Corollary 2.6.8). It follows that, over a ring k of characteristic p, the ring stack $\mathbb{G}_a^{\mathrm{dR}}$ is a square-zero extension of $F_*\mathbb{G}_a$ by $\mathrm{B}F_*\mathbb{G}_a^{\#}$ (Corollary 2.6.11). Via transmutation, we see a map $\nu : (X/k)^{\mathrm{dR}} \to X^{(1)}$ that realizes the source as a $\mathrm{T}^{\#}_{X^{(1)}/k}$-gerbe (Proposition 2.7.1). Then the conjugate filtration of the de Rham cohomology of X is understood as that induced by the canonical filtration of $\nu_*\mathcal{O}$; in fact there is a natural isomorphism $\nu_*\mathcal{O} \simeq F_{X/k,*}\Omega^{\bullet}_{X/k}$ over $X^{(1)}$, which is an $\mathcal{O}_{X^{(1)}}$-linear refinement of the isomorphism proved in Talk 3. Prove an isomorphism $\mathrm{H}^*(\nu_*\mathcal{O}) \simeq \Omega^*_{X^{(1)}/k}$ of graded algebras over $X^{(1)}$ (Corollary 2.7.2), which explains the classical Cartier isomorphism.

Talk 5 (Deligne–Illusie theorem [§2.7.2]). This talk begins with a conjugate-filtered refinement of the Witt vector model for $\mathbb{G}_a^{\mathrm{dR}}$ explained in Talk 4. We work over a ring k of characteristic p. Introduce the filtered ring stack $\mathbb{G}_a^{\mathrm{dR},c}$ with the underlying ring stack $\mathbb{G}_a^{\mathrm{dR}}$ (Construction 2.7.8). Explain that, for a smooth qcqs k-scheme X, the cohomology of the filtered stack $(X/k)^{\mathrm{dR},c}$ defined via transmutation from $\mathbb{G}_a^{\mathrm{dR},c}$ recovers the conjugate-filtered de Rham cohomology of X (Theorem 2.7.9). Give the Witt vector model for $\mathbb{G}_a^{\mathrm{dR},c}$ (Construction 2.7.11 and Proposition 2.7.12). Through this model, we see a $\mathbb{G}_m^{\#}$-action on the filtered ring stack $\mathbb{G}_a^{\mathrm{dR},c}$, in fact, an action as a filtered W/p^2-algebra stack (Remark 2.7.13).

Prove the Deligne–Illusie theorem using this $\mathbb{G}_m^{\#}$-action as in Corollary 2.7.14. Discuss other recent developments around Deligne–Illusie, such as [Pet23, Pet25]. In particular, the ring stack $\mathbb{G}_a^{\mathrm{dR}}$ plays an essential role in the proof of the main theorem in [Pet25] (see also Remark 2.7.5 and Corollary 2.7.6).

Talk 6 (Filtered prismatization in characteristic p [§3.1, §3.2, §3.3]). We work over a perfect filed k of characteristic $p > 0$. Recall the crystallization which is (implicitly) discussed in Talk 3 (Remark 2.5.12). The prismatization is its Frobenius twist, and the underlying formal ring stack is simply W/p (Construction 3.1.1).

The aim of this talk is to present the Nygaard filtration on the prismatic cohomology from the stacky point of view. The Nygaard filtration on W is the p-adic filtration by definition, and note that it corresponds to the graded $W[t]$-algebra $W[u,t]/(ut-p)$ under the Rees equivalence. The formal stack $k^{\mathcal{N}}$ is defined to be $\mathrm{Spf}(W[u,t]/(ut-p))/\mathbb{G}_m$ and is called the *filtered prismatization* of k (Construction 3.3.1). Introduce the ring stack $\mathbb{G}_a^{\mathcal{N}}$ over $k^{\mathcal{N}}$ and note that it naturally has a W/p-algebra structure (Construction 3.3.2). Hence, given a k-scheme X, its filtered prismatization $X^{\mathcal{N}}$ is well-defined as a stack over $k^{\mathcal{N}}$ via transmutation from $\mathbb{G}_a^{\mathcal{N}}$. Prove that this lifts the Hodge and conjugate filtered de Rhamification, as well as the crystallization and prismatization (Theorem 3.3.5).

The direct image of the structure sheaf under $X^{\mathcal{N}} \to k^{\mathcal{N}}$ is regarded as a p^*W-module in the p-complete filtered W-modules whose underlying W-module is the (Frobenius-twisted) prismatic cohomology of X; which is called the *Nygaard filtration*. Via transmutation, the W/p-algebra structure on $\mathbb{G}_a^{\mathcal{N}}$ gives the prismatic Frobenius map from the Nygaard filtration to the p-adic filtration on the prismatic cohomology. Explain that the Nygaard graded pieces are identified with the conjugate filtrations of the de Rham cohomology under the prismatic Frobenius and that it characterizes the Nygaard filtration (Theorem 3.2.1 and Remark 3.2.3).

To be continued in the fall.

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