RELATIVE K_0 AND RELATIVE CYCLE CLASS MAP

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Abstract. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. We study the zeroth homotopy group $K_0(F)$ of the homotopy fiber of the map $K(\mathcal{A}) \to K(\mathcal{B})$ between K-theory spectra. Under the assumption that F is a cofinal and that $\mathcal{B}$ is split exact, we give an explicit description of $K_0(F)$ in terms of the triangulated functor $\mathsf{D}^b(\mathcal{A}) \to \mathsf{D}^b(\mathcal{B})$ between the derived categories.

We apply it to the pair (X, D) of a scheme X and an affine closed subscheme D of X, and get a description of the relative K_0-group $K_0(X, D)$ in terms of perfect complexes; it is generated by pairs of two perfect complexes of X together with quasi-isomorphisms along D. This description makes it possible to assign a cycle class in $K_0(X, D)$ to a cycle on X not meeting D in an intuitive way. When X is a separated regular scheme of finite type over a field and D is an affine effective Cartier divisor on X, we prove that the cycle classes induce a surjective group homomorphism from the Chow group with modulus $\mathrm{CH}_*(X|D)$ defined by Binda-Saito to a suitable subquotient of $K_0(X, D)$.

Contents

0. Introduction

0.1. Let X be a separated regular scheme of finite type over a field. Then, to every integral closed subscheme V of X, we can assign the cycle class $\mathsf{cyc}(V)$ in the Grothendieck group $K_0(X)$ of algebraic vector bundles on X. Grothendieck has shown that the cycle classes induce surjective group homomorphisms from the Chow groups to subquotients of $K_0(X)$

$$\mathsf{cyc}\colon\ \mathrm{CH}_k(X) \twoheadrightarrow F_k K_0(X)/F_{k-1}K_0(X) \tag{0.1}$$

for all $k \geq 0$, cf. [SGA6, Exp 0, App. Ch II]. Here, F_* is the coniveau filtration; $F_k K_0(X)$ is generated by perfect complexes of X whose supports are of dimension $\leq k$.

The current paper constructs a relative version of the cycle class map (0.1). Let D be an effective Cartier divisor on X. We are interested in the relative K_0-group $K_0(X, D)$, which is defined to be the zeroth homotopy group of the homotopy fiber of the canonical map $K(X) \to K(D)$ between K-theory spectra. As a cycle theoretical invariant, we use the Chow group with modulus $\mathrm{CH}_*(X|D)$ defined by Binda-Saito [BS17]: It is the group generated by cycles on X which do not meet D divided by a variant of rational equivalence (see §3.3 for details). Here is the main theorem, which generalizes (0.1).

Theorem 0.1 (Theorem 3.7, Lemma 3.4)**.** *Let X be a separated regular scheme of finite type over a field and D an affine effective Cartier divisor on X. Then there exist surjective group homomorphisms*

$$\mathsf{cyc}\colon\ \mathrm{CH}_k(X|D) \twoheadrightarrow F_k K_0(X, D)/F_{k-1}K_0(X, D) \tag{0.2}$$

for all $k \geq 0$, where F_* is the "coniveau filtration" (Definition 3.3). Furthermore, if D has an affine open neighborhood in X, then $F_{\dim X} K_0(X, D) = K_0(X, D)$.

In [BK18], Binda-Krishna constructed a cycle class map for zero cycles with modulus, namely a map from $\mathrm{CH}_0(X|D)$ to $K_0(X,D)$, for modulus pairs (X,D) with X smooth quasi-projective over a perfect field. They have also shown that the cycle class map is injective if X is affine and the base filed is algebraically closed. Also, Binda [Bi18] constructed a cycle class map for higher zero cycles with modulus using a slightly different (stronger) modulus condition.

If X is a smooth quasi-projective scheme over a field, then Grothendieck's Riemann-Roch type formula implies that the cycle class map (0.1) is a rational isomorphism. In a subsequent paper [IK18], we prove that the relative cycle map (0.2) is a rational isomorphism, at least when X is smooth affine.

0.2. Let X be a separated regular noetherian scheme and V an integral closed subscheme of X. Then the coherent sheaf $\mathcal{O}_V$ is a perfect complex of X, i.e. quasi-isomorphic to a bounded complex $E_\bullet$ of algebraic vector bundles on X, and the cycle class is given by $\mathsf{cyc}(V) := \sum(-1)^i[E_i] \in K_0(X)$. Here, it is more natural to consider the group $K_0^{\mathrm{perf}}(X)$ generated by perfect complexes of X with the relation $[P] = [P'] + [P'']$ for each exact triangle $P' \to P \to P'' \to P'[1]$. It follows from [SGA6, Exp 1, 6.4] that the canonical map

$$K_0(X) \xrightarrow{\simeq} K_0^{\mathrm{perf}}(X) \tag{0.3}$$

is an isomorphism. Under this isomorphism, the cycle class $\mathsf{cyc}(V)$ is just the class of the perfect complex $\mathcal{O}_V$ in $K_0^{\mathrm{perf}}(X)$.

Now, suppose we are given an affine closed subscheme D of X, and we denote the inclusion $D \hookrightarrow X$ by ι. Then the relative K_0-group $K_0(X,D)$ is generated by pairs (E,E') of algebraic vector bundles on X together with isomorphisms $E|_D \xrightarrow{\simeq} E'|_D$ along D (Theorem 1.5). As in the absolute case, perfect complexes are more appropriate to construct cycle classes. We define $K_0^{\mathrm{perf}}(X,D)$ to be the group generated by pairs (P,P') of perfect complexes of X together with isomorphisms $L\iota^*P \xrightarrow{\simeq} L\iota^*P'$ in the derived category of D with suitable relations (Definition 2.1). Then we show that the canonical map

$$K_0(X,D) \xrightarrow{\simeq} K_0^{\mathrm{perf}}(X,D) \tag{0.4}$$

is an isomorphism (Theorem 3.1). This is a generalization of (0.3).

Let V be an integral closed subscheme of X which does not meet D. Then $\mathcal{O}_V$ is a perfect complex of X and $L\iota^*\mathcal{O}_V \simeq 0$. Hence, the pair $(\mathcal{O}_V, 0)$ gives an element of $K_0^{\mathrm{perf}}(X,D)$, which we denote by $\mathsf{cyc}(V)$. When X is of finite type over a field and D is a Cartier divisor, we show that cyc kills the relations that define $\mathrm{CH}_*(X|D)$ and get Theorem 0.1.

The hardest part of the above argument is the proof of the isomorphism (0.4). This isomorphism holds more generally for a certain type of exact functors between small exact categories. Actually, in large part of this paper, we discuss relative K-theory of exact categories and triangulated categories in general.

Here is a brief summary of the contents of this paper. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. We define $K_0(F)$ to be the group generated by pairs (P,Q) of two objects in $\mathcal{A}$ together with isomorphisms $F(P) \xrightarrow{\simeq} F(Q)$ in $\mathcal{B}$ with suitable relations (Definition 1.2). Then, under the assumption that $\mathcal{B}$ is split exact and that F is cofinal, $K_0(F)$ is isomorphic to the zeroth homotopy group of the homotopy fiber of the map $K(\mathcal{A}) \to K(\mathcal{B})$ between the K-theory spectra (Theorem 1.5). This essentially follows from Heller's result in [He65] and we explain it in the first section §1. The second section §2 is the technical heart of this paper. We define a group $K_0(T)$ for a triangulated functor T between small triangulated categories (Definition 2.1) as an analogue of the K_0 for an exact functor between small exact categories. Then, for an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between small exact categories with $\mathcal{B}$ being split exact, we prove that $K_0(F)$ is isomorphic to the K_0 of the triangulated functor $\mathsf{D}^b(\mathcal{A}) \to D^b(\mathcal{B})$ between the derived categories (Theorem 2.4). This is the general assertion of the isomorphism (0.4). Finally, in the third section §3, we prove Theorem 0.1.

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1. Relative K_0 of exact categories

Let us start from a general construction of a category, which is used throughout this paper.

Definition 1.1. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of categories. We define a category $\mathsf{Rel}(F)$:

- Objects are triples (P, α, Q) with $P, Q \in \mathcal{A}$ and $\alpha\colon F(P) \xrightarrow{\simeq} F(Q)$ an isomorphism in $\mathcal{B}$.
- Morphisms from (P, α, Q) to (P', α', Q') are pairs (f, g) of morphisms $f\colon P \to P'$ and $g\colon Q \to Q'$ in $\mathcal{A}$ which make the diagram

$$\begin{array}{ccc} F(P) & \xrightarrow{F(f)} & F(P') \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ F(Q) & \xrightarrow{F(g)} & F(Q') \end{array}$$

commutative.

1.1. **Heller's theorem.**

Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between exact categories. We call a sequence

$$(P', \alpha', Q') \xrightarrow{(f,g)} (P, \alpha, Q) \xrightarrow{(f',g')} (P'', \alpha'', Q'')$$

in $\mathsf{Rel}(F)$ *exact* if $P' \xrightarrow{f} P \xrightarrow{f'} P''$ and $Q' \xrightarrow{g} Q \xrightarrow{g'} Q''$ are exact sequences in $\mathcal{A}$. Under this definition, $\mathsf{Rel}(F)$ is an exact category.

Definition 1.2. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. We define $K_0(F)$ to be the group with the generators $[X]$, one for each $X \in \mathsf{Rel}(F)$, and with the following relations:

(a) For each exact sequence $X' \rightarrowtail X \twoheadrightarrow X''$ in $\mathsf{Rel}(F)$,
$$[X] = [X'] + [X''].$$
(b) For each pair $((P, \alpha, Q), (Q, \beta, R))$ of objects in $\mathsf{Rel}(F)$,
$$[(P, \alpha, Q)] + [(Q, \beta, R)] = [(P, \beta\alpha, R)].$$

Definition 1.3. An exact category is *split exact* if every exact sequence is split exact.

Definition 1.4. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between additive category is *cofinal* if for every $B \in \mathcal{B}$ there exists $B' \in \mathcal{B}$ and $A \in \mathcal{A}$ such that $F(A) \simeq B \oplus B'$.

For a small exact category $\mathcal{A}$, we denote Quillen's K-theory spectrum by $K(\mathcal{A})$. Here is a reinterpretation of Heller's result in [He65].

Theorem 1.5. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Suppose that $\mathcal{B}$ is split exact and that F is cofinal. Then there exists a natural isomorphism of groups*

$$K_0(F) \simeq \pi_0 \operatorname{hofib}(K(\mathcal{A}) \xrightarrow{F} K(\mathcal{B})).$$

We give a proof in §1.4.

1.2. **Basic properties.**

Here, we collect some basic properties of relative K_0-groups of exact categories (Definition 1.2), whose proof is immediate from the definition.

Lemma 1.6. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Then:*

(i) $[0] = 0$ *in* $K_0(F)$. *If* $X, Y \in \mathsf{Rel}(F)$ *are isomorphic, then* $[X] = [Y]$ *in* $K_0(F)$.
(ii) *If* $\gamma\colon P \xrightarrow{\simeq} Q$ *is an isomorphism in* $\mathcal{A}$*, then* $[(P, F(\gamma), Q)] = 0$ *in* $K_0(F)$.
(iii) *For every* $(P, \alpha, Q) \in \mathsf{Rel}(F)$, $[(P, \alpha, Q)] + [(Q, \alpha^{-1}, P)] = 0$ *in* $K_0(F)$.
(iv) *Every element of* $K_0(F)$ *has the form* $[X]$ *for some* $X \in \mathsf{Rel}(F)$.

Definition 1.7. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Given two objects $(P, \alpha, Q), (P', \alpha', Q')$ in $\mathsf{Rel}(F)$, we write

$$(P', \alpha', Q') \looparrowright (P, \alpha, Q)$$

if there exist $N \in \mathcal{A}$ and a commutator γ in $\mathrm{Aut}(F(Q))$ which fit into an exact sequence

$$(P', \alpha', Q') \rightarrowtail (P, \gamma\alpha, Q) \twoheadrightarrow (N, 1, N).$$

Lemma 1.8. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories and $X, X' \in \mathsf{Rel}(F)$. If $X' \looparrowright X$, then $[X'] = [X]$ in $K_0(F)$.*

Remark 1.9. In fact, if $\mathcal{B}$ is split exact, then the converse holds, i.e. all relations of $K_0(F)$ are generated by $\looparrowright$ (and $\looparrowleft$).

1.3. **Elementary transformations.** Let $\mathcal{C}$ be an additive category. Let $P, Q \in \mathcal{C}$. Suppose that P and Q have the forms $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$. Then a homomorphism from P to Q can be expressed by a matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : P_1 \oplus P_2 \to Q_1 \oplus Q_2,$$

where a_{ij} is a morphism $P_j \to Q_i$ in $\mathcal{C}$.

Definition 1.10. An endomorphism α of $P \in \mathcal{C}$ is an *elementary transformation* if there exists an embedding $\mathcal{C} \hookrightarrow \overline{\mathcal{C}}$ of additive categories and α is isomorphic to an endomorphism of $P_1 \oplus P_2$ of the form

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

for some $P_1, P_2 \in \overline{\mathcal{C}}$ and $a\colon P_2 \to P_1$. We denote by $E(P)$ the subgroup of $\mathrm{Aut}(P)$ generated by elementary transformations.

Lemma 1.11. *Let $P \in \mathcal{C}$ and $\alpha \in E(P)$. Then*

$$\alpha \oplus 1\colon P \oplus P \to P \oplus P$$

is a commutator of $\mathrm{Aut}(P \oplus P)$.

Proof. We may assume that α is an elementary transformation, i.e. $\exists \beta\colon P \xrightarrow{\simeq} P_1 \oplus P_2$ and

$$\beta\alpha\beta^{-1} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

for some $a\colon P_2 \to P_1$. Then

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a commutator of $P_1 \oplus P_2 \oplus P_2$. Indeed, this is equal to

$$\left[\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right]$$

in $\mathrm{Aut}(P_1 \oplus P_2 \oplus P_2)$. This implies that $\alpha \oplus 1\colon P \oplus P \to P \oplus P$ is a commutator of $\mathrm{Aut}(P \oplus P)$. □

Corollary 1.12. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Let $(P, \alpha, Q) \in \mathsf{Rel}(F)$ and $\gamma \in E(F(Q))$. Then*

$$[(P, \alpha, Q)] = [(P, \gamma\alpha, Q)]$$

in $K_0(F)$.

Proof. According to Lemma 1.11,

$$(P, \alpha, Q) \looparrowright (P \oplus Q, \alpha \oplus 1, Q \oplus Q) \looparrowleft (P, \gamma\alpha, Q).$$

Hence, $[(P, \alpha, Q)] = [(P, \gamma\alpha, Q)]$ by Lemma 1.8. □

1.4. **Proof of Theorem 1.5.** Let $\mathcal{A}$ be a small exact category. In [Ne98], Nenashev provides generators and relations for $K_1(\mathcal{A})$; the generators are double exact sequences. In case $\mathcal{A}$ is split exact (Definition 1.3), the generators and relations become simpler and the resulting group coincides with the one considered by Heller in [He65].

Lemma 1.13. *Let $\mathcal{A}$ be a small exact category. We define $K_1^{\mathrm{He}}(\mathcal{A})$ to be the abelian group with the generators $[h]$, one for each $P \in \mathcal{A}$ and each $h \in \mathrm{Aut}(P)$, and with the following relations:*

(a) *For a commutative diagram*

$$\begin{array}{ccccc} P' & \overset{f}{\rightarrowtail} & P & \overset{g}{\twoheadrightarrow} & P'' \\ \simeq \downarrow h' & & \simeq \downarrow h & & \simeq \downarrow h'' \\ P' & \overset{f}{\rightarrowtail} & P & \overset{g}{\twoheadrightarrow} & P'' \end{array}$$

with exact rows, $[h] = [h'] + [h'']$.

(b) *$[h_2 \circ h_1] = [h_2] + [h_1]$ for $h_1, h_2 \in \mathrm{Aut}(P)$.*

If $\mathcal{A}$ is split exact, then there exists a natural isomorphism

$$K_1^{\mathrm{He}}(\mathcal{A}) \xrightarrow{\simeq} K_1(\mathcal{A}).$$

Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. We suppose that F is cofinal (Definition 1.3). Then every element of $K_1^{\mathrm{He}}(\mathcal{B})$ is represented by $g \in \mathrm{Aut}(F(P))$ for some $P \in \mathcal{A}$. According to [He65, Proposition 4.2], the class $[(P, g, P)]$ in $K_0(F)$ does not depend on the representative and gives a group homomorphism

$$\delta\colon K_1^{\mathrm{He}}(\mathcal{B}) \to K_0(F).$$

We define a map $\iota\colon K_0(F) \to K_0(\mathcal{A})$ by sending $[(P, \alpha, Q)]$ to $[P] - [Q]$.

Proposition 1.14 (Heller)**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between small exact categories. Suppose that $\mathcal{B}$ is split exact and that F is cofinal. Then the sequence*

$$K_1(\mathcal{A}) \xrightarrow{F} K_1(\mathcal{B}) \xrightarrow{\delta} K_0(F) \xrightarrow{\iota} K_0(\mathcal{A}) \xrightarrow{F} K_0(\mathcal{B}).$$

is exact.

Proof. Heller showed the sequence

$$K_1^{\mathrm{He}}(\mathcal{A}) \xrightarrow{F} K_1^{\mathrm{He}}(\mathcal{B}) \xrightarrow{\delta} K_0(F) \xrightarrow{\iota} K_0(\mathcal{A}) \xrightarrow{F} K_0(\mathcal{B}).$$

is exact [He65, Proposition 5.2]. Now, $K_1^{\mathrm{He}}(\mathcal{B}) = K_1(\mathcal{B})$ and the map $K_1^{\mathrm{He}}(\mathcal{A}) \to K_1^{\mathrm{He}}(\mathcal{B})$ factors through $K_1(\mathcal{A})$. This proves the exactness at $K_1(\mathcal{B})$. □

Proof of Theorem 1.5. In [GG87], Gillet and Grayson have constructed a simplicial set $G\mathcal{A}$ such that its geometric realization is naturally homotopy equivalent to $K(\mathcal{A})$. The 0-simplexes of $G\mathcal{A}$ are pairs (P, Q) of objects in $\mathcal{A}$. The 1-simplexes of $G\mathcal{A}$ are pairs of exact sequences of the forms

$$P_0 \rightarrowtail P_1 \twoheadrightarrow P_{01}\,, \quad Q_0 \rightarrowtail Q_1 \twoheadrightarrow P_{01}\,.$$

The face maps $G\mathcal{A}_1 \to G\mathcal{A}_0$ send the above to (P_0, Q_0) and (P_1, Q_1) respectively.

Let GF_0 be the set of objects in $\mathsf{Rel}(F)$. Let GF_1 be the set of all pairs (l, γ) where l is an exact sequence in $\mathsf{Rel}(F)$ of the form

$$l\colon \quad (P, \alpha, Q) \rightarrowtail (R, \beta, S) \twoheadrightarrow (N, 1, N)$$

and γ is a commutator of $\mathrm{Aut}(F(S))$. We define face maps $d_1, d_2\colon GF_1 \to GF_0$ by $d_1((l, \gamma)) := (P, \alpha, Q)$ and $d_2((l, \gamma)) := (R, \gamma\beta, S)$, and a degeneracy map $s\colon GF_0 \to GF_1$ by $s(X) := (X \xrightarrow{1} X \to 0, 1)$. We define GF to be the simplicial set generated by GF_1, GF_0. Then

$$\pi_0|GF| = K_0(F).$$

We have a natural map $GF \to G\mathcal{A}$ which sends $(P, \alpha, Q) \in GF_0$ to (P, Q) and $(l, \gamma) \in GF_1$ to the underlying pair of exact sequences of l. Then the composite $GF \to G\mathcal{A} \to G\mathcal{B}$ is homotopic to zero. Therefore, we obtain a natural map

$$\theta\colon K_0(F) \to \pi_0 \operatorname{hofib}(K(\mathcal{A}) \to K(\mathcal{B})).$$

According to Proposition 1.14, it remains to show that $\delta\colon K_1(\mathcal{B}) \to K_0(F)$ followed by θ is equal to the boundary map

$$\partial\colon K_1(\mathcal{B}) \to \pi_0 \operatorname{hofib}(K(\mathcal{A}) \to K(\mathcal{B})),$$

and it is straightforward. □

Remark 1.15. The construction of GF_0 and GF_1 in the proof suggests that there might be an algebraic construction of $GF_2, GF_3, \dots$ so that the resulting simplicial set is a model of the double loop space of the relative Waldhausen construction $wS_\bullet(S_\bullet F)$ in, say, [We13, 8.5.3].

Example 1.16. Here are examples of exact functors $F\colon \mathcal{A} \to \mathcal{B}$ between exact categories such that $\mathcal{B}$ is split exact and that F is cofinal.

(1) A base change functor $\mathbf{P}(A) \to \mathbf{P}(B)$ induced from a ring homomorphism $A \to B$. Here, $\mathbf{P}(-)$ is the category of finitely generated projective modules.
(2) A base change functor $\mathsf{Vec}(X) \to \mathsf{Vec}(Y)$ induced from a morphism of schemes $Y \to X$ with Y affine. Here, $\mathsf{Vec}(-)$ is the category of algebraic vector bundles.

2. Relative K_0 of triangulated categories

The goal in this section is to prove Theorem 2.4.

2.1. **The definition and basic properties.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a triangulated functor between triangulated categories. Refer to Definition 1.1 for the definition of the category $\mathsf{Rel}(F)$. We denote by $[1]$ the shift functor of $\mathcal{A}$ or $\mathcal{B}$, and we define an endofunctor of $\mathsf{Rel}(F)$ by $(P, \alpha, Q)[1] := (P[1], \alpha[1], Q[1])$. We call a sequence

$$(P_1, \alpha_1, Q_1) \xrightarrow{(f_1, g_1)} (P_2, \alpha_2, Q_2) \xrightarrow{(f_2, g_2)} (P_3, \alpha_3, Q_3) \xrightarrow{(f_3, g_3)} (P_1, \alpha_1, Q_1)[1]$$

in $\mathsf{Rel}(F)$ an *exact triangle* if

$$P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3 \xrightarrow{f_3} P_1[1]$$

$$Q_1 \xrightarrow{g_1} Q_2 \xrightarrow{g_2} Q_3 \xrightarrow{g_3} Q_1[1]$$

are exact triangles in $\mathcal{A}$. Under this definition, $\mathsf{Rel}(F)$ is an additive category which satisfies the first two axioms (TR1) and (TR2) of triangulated category in [Ve77, 1.1], but may not satisfy the other axioms (TR3) nor (TR4).

Definition 2.1. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a triangulated functor between small triangulated categories. We define $K_0(F)$ to be the group with the generators $[X]$, one for each $X \in \mathsf{Rel}(F)$, and with the following relations:

(a) For each exact triangle $X' \to X \to X'' \to X'[1]$ in $\mathsf{Rel}(F)$,
$$[X] = [X'] + [X''].$$
(b) For each pair $((P, \alpha, Q), (Q, \beta, R))$ of objects in $\mathsf{Rel}(F)$,
$$[(P, \alpha, Q)] + [(Q, \beta, R)] = [(P, \beta\alpha, R)].$$

The same properties as in Lemma 1.6 also hold for triangulated categories. Again, the proof is immediate.

Lemma 2.2. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a triangulated functor between small triangulated categories. Then:*

(i) $[0] = 0$ *in $K_0(F)$. If $X, Y \in \mathsf{Rel}(F)$ are isomorphic, then $[X] = [Y]$ in $K_0(F)$.*
(ii) *For every $X \in \mathsf{Rel}(F)$ and every integer n, $[X[n]] = (-1)^n[X]$ in $K_0(F)$.*

(iii) *If* $\gamma\colon P \xrightarrow{\simeq} Q$ *is an isomorphism in* $\mathcal{A}$*, then* $[(P, F(\gamma), Q)] = 0$ *in* $K_0(F)$*.*
(iv) *For every* $(P, \alpha, Q) \in \mathsf{Rel}(F)$*,* $[(P, \alpha, Q)] + [(Q, \alpha^{-1}, P)] = 0$ *in* $K_0(F)$*.*
(v) *Every element of* $K_0(F)$ *has the form* $[X]$ *for some* $X \in \mathsf{Rel}(F)$*.*

Lemma 2.3. *Let* $F\colon \mathcal{A} \to \mathcal{B}$ *be a triangulated functor between small triangulated categories. Let* $\mathcal{A}_0$ *be a thick triangulated subcategory of* $\mathcal{A}$ *such that* $F(\mathcal{A}_0) = 0$*. Then* F *factors through the Verdier quotient* $\mathcal{A}/\mathcal{A}_0$*, and there is an exact sequence of abelian groups*

$$K_0(\mathcal{A}_0) \longrightarrow K_0(\mathcal{A} \xrightarrow{F} \mathcal{B}) \longrightarrow K_0(\mathcal{A}/\mathcal{A}_0 \xrightarrow{\bar{F}} \mathcal{B}) \longrightarrow 0.$$

Proof. Note that $\mathsf{Ob}(\mathsf{Rel}(F)) = \mathsf{Ob}(\mathsf{Rel}(\bar{F}))$. An exact triangle in $\mathsf{Rel}(\bar{F})$ is a sequence

$$(P_1, \alpha_1, Q_1) \xrightarrow{(f_1, g_1)} (P_2, \alpha_2, Q_2) \xrightarrow{(f_2, g_2)} (P_3, \alpha_3, Q_3) \xrightarrow{(f_3, g_3)} (P_1, \alpha_1, Q_1)[1]$$

in $\mathsf{Rel}(\bar{F})$ such that

$$P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3 \xrightarrow{f_3} P_1[1]$$
$$Q_1 \xrightarrow{g_1} Q_2 \xrightarrow{g_2} Q_3 \xrightarrow{g_3} Q_1[1]$$

are isomorphic in $\mathcal{A}/\mathcal{A}_0$ to exact triangles in $\mathcal{A}$. It follows that $K_0(\bar{F})$ is the group with the generators $[X]$, one for each $X \in \mathsf{Rel}(F)$, and with the relations (a) (b) of Definition 2.1 and an additional relation

(c) For $X = (P, 0, Q)$ with $P, Q \in \mathcal{A}_0$, $[X] = 0$.

This says that $K_0(\bar{F})$ is the quotient of $K_0(F)$ by the image of $K_0(\mathcal{A}_0)$. □

2.2. **Comparison theorem.** For an additive category $\mathcal{A}$, we use the following notation:

(1) $\mathsf{Ch}^b(\mathcal{A})$ is the category of bounded chain complexes in $\mathcal{A}$.
(2) $\mathsf{K}^b(\mathcal{A})$ is the bounded homotopy category, i.e. the same object with $\mathsf{Ch}^b(\mathcal{A})$ and morphisms up to homotopy. We regard $\mathsf{K}^b(\mathcal{A})$ as a triangulated category in the standard way (cf. [Ve77]).

Here is the main theorem in this section.

Theorem 2.4. *Let* $\mathcal{A}$ *be a small exact category which is closed under the kernels of surjections. Let* $\mathcal{B}$ *be a small split exact category and* $F\colon \mathcal{A} \to \mathcal{B}$ *an exact functor. We define* $\mathsf{K}^{b,\emptyset}(\mathcal{A})$ *to be the full subcategory of* $\mathsf{K}^b(\mathcal{A})$ *consisting acyclic complexes. Then* F *induces a triangulated functor*

$$\mathsf{D}(F)\colon \mathsf{K}^b(\mathcal{A})/\mathsf{K}^{b,\emptyset}(\mathcal{A}) \to \mathsf{K}^b(\mathcal{B})$$

and the canonical map

$$K_0(F) \xrightarrow{\simeq} K_0(\mathsf{D}(F))$$

is an isomorphism.

Here, we have chosen an embedding of $\mathcal{A}$ into an abelian category (such an embedding exists by Freyd-Mitchell theorem). The terms “kernels of surjections” and “acyclic complexes” are understood in this abelian category. The theorem implies that $K_0(\mathsf{D}(F))$ does not depend on the choice of the embedding.

The difficulty in the proof of Theorem 2.4 is how to define the inverse. In absolute case, i.e. $\mathcal{B} = 0$, the inverse is clear, which is given by $E_\bullet \mapsto \sum(-1)^i[E_i]$. We cannot imitate this directly because we have to keep track of homotopy equivalences in $\mathcal{B}$. However, a variant does still work. We call it the Euler characteristic and study in §2.3 and §2.4. Using this machinery, the proof of Theorem 2.4 is completed in §2.5.

We fix $\mathcal{A}$, $\mathcal{B}$ and F as in Theorem 2.4 until the end of this section.

2.3. **Euler characteristic I.** Let C be a bounded complex in $\mathcal{B}$ which is homotopic to zero. Then there exists $s\colon C_n \to C_{n+1}$ such that $sd + ds = 1$ and $ss = 0$, which we call a *strict split* of C. Then the direct sum of the maps

$$\begin{array}{ccc} & & C_{n+1} \\ & \overset{s}{\nearrow} & \\ C_n & & \\ & \underset{d}{\searrow} & \\ & & C_{n-1}, \end{array}$$

gives an isomorphism

$$\Phi_{C,s}\colon \bigoplus_n C_{2n+1} \xrightarrow{\simeq} \bigoplus_n C_{2n}.$$

Lemma 2.5. *If s' is another strict split of C, then there exists an elementary transformation (Definition 1.10) γ of C_{2*} such that $\Phi_{C,s} = \gamma\Phi_{C,s'}$.*

Proof. Take an embedding of $\mathcal{B}$ into an abelian category $\overline{\mathcal{B}}$ by Freyd-Mitchell theorem. Let $Z_n \in \overline{\mathcal{B}}$ be the kernel of $d\colon C_n \to C_{n-1}$. Then C decomposes into short exact sequences

$$0 \longrightarrow Z_n \xrightarrow{\epsilon} C_n \xrightarrow{\delta} Z_{n-1} \longrightarrow 0,$$

and $s\epsilon\colon Z_{n-1} \to C_n$ or $\delta s\colon C_n \to Z_n$ give splits of these short exact sequences. Hence, the map

$$\phi_{s,n} := (\delta s, \delta)\colon C_n \xrightarrow{\simeq} Z_n \oplus Z_{n-1},$$

is an isomorphism with the inverse $\phi_{s,n}^{-1} = (\epsilon, s\epsilon)$.

Now, it is clear that there is an elementary transformation γ_n of $Z_n \oplus Z_{n-1}$ such that $\phi_{s,n} = \gamma_n\phi_{s',n}$. On the other hand, $\phi_{s,n}$'s give isomorphisms

$$C_{2*+1} \underset{\simeq}{\xrightarrow{\phi_{s,2*+1}}} Z_* \underset{\simeq}{\xrightarrow{\phi_{s,2*}^{-1}}} C_{2*},$$

and the composite is equal to $\Phi_{C,s}$ because $\epsilon\delta = d$ and $s\epsilon\delta s = sds = s$. Therefore, there exists an elementary transformation γ such that $\Phi_{C,s} = \gamma\Phi_{C',s}$. ☐

Let P, Q be bounded complexes in $\mathcal{A}$ and α a homotopy equivalence $F(P) \xrightarrow{\sim} F(Q)$. We apply the above construction to $\operatorname{cone}\alpha$. Set $\Phi := \Phi_{\operatorname{cone}\alpha,s}$ for some strict split s of $\operatorname{cone}\alpha$. Thanks to Lemma 2.5 and Corollary 1.12, the following is well-defined.

Definition 2.6. We define the *Euler characteristic of* (P, α, Q) by

$$\chi(P, \alpha, Q) := \Big[\Big(\bigoplus_n (P_{2n} \oplus Q_{2n+1}), \Phi, \bigoplus_n (P_{2n-1} \oplus Q_{2n})\Big)\Big] \in K_0(F).$$

Here are first properties of the Euler characteristic.

Lemma 2.7. *Let P, Q be bounded complexes in $\mathcal{A}$ with a homotopy equivalence $\alpha\colon F(P) \xrightarrow{\sim} F(Q)$.*

(i) *Let $P', Q' \in \mathsf{Ch}^b(\mathcal{A})$ with isomorphisms of complexes $\gamma\colon F(P) \xrightarrow{\simeq} F(P')$ and $\delta\colon F(Q) \xrightarrow{\simeq} F(Q')$. If $[(P_n, \gamma_n, P'_n)] = [(Q_n, \delta_n, Q'_n)] = 0$ for all n, then*
$$\chi(P', \delta\alpha\gamma^{-1}, Q') = \chi(P, \alpha, Q).$$

(ii) *For every integer n, $\chi(P[n], \alpha[n], Q[n]) = (-1)^n\chi(P, \alpha, Q)$.*

Proof. (i) $\gamma := (\gamma, \delta)$ gives an isomorphism of complexes $\operatorname{cone}\alpha \xrightarrow{\simeq} \operatorname{cone}(\delta\alpha\gamma^{-1})$. Hence, $\Phi_{\operatorname{cone}(\delta\alpha\gamma^{-1})}$ is equal to the composite

$$F(P'_{2*}) \oplus F(Q'_{2*+1}) \xrightarrow{(\gamma_{2*}^{-1}, \delta_{2*+1}^{-1})} F(P_{2*}) \oplus F(Q_{2*+1})$$
$$\xrightarrow{\Phi_{\operatorname{cone}\alpha}} F(P_{2*-1}) \oplus F(Q_{2*}) \xrightarrow{(\gamma_{2*-1}, \delta_{2*})} F(P'_{2*-1}) \oplus F(Q'_{2*}).$$

It follows from the assumption that

$$(P'_{2*} \oplus Q'_{2*+1}, (\gamma_{2*}^{-1}, \delta_{2*+1}^{-1}), P_{2*} \oplus Q_{2*+1}) = (P_{2*-1} \oplus Q_{2*}, (\gamma_{2*-1}, \delta_{2*}), P'_{2*-1} \oplus Q'_{2*}) = 0,$$

and thus $\chi(P', \delta\alpha\gamma^{-1}, Q') = \chi(P, \alpha, Q)$.

(ii) By the construction,

$$\chi(P[1], \alpha[1], Q[1]) = [(P_{2*} \oplus Q_{2*+1}, -\Phi^{-1}_{\mathrm{cone}(-\alpha)}, P_{2*-1} \oplus Q_{2*})].$$

By Lemma 2.2 (iv), the right hand side equals to

$$-[(P_{2*-1} \oplus Q_{2*}, \Phi_{\mathrm{cone}(-\alpha)}, P_{2*} \oplus Q_{2*+1})] = -\chi(P, -\alpha, Q),$$

which equals to $-\chi(P, \alpha, Q)$ by (i). □

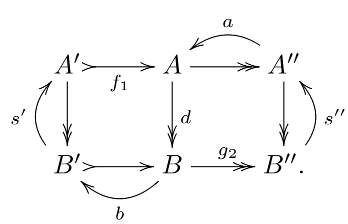
We will use the following lemma on homological algebra.

Lemma 2.8. *Let $\mathcal{C}$ be an additive category. Suppose given a commutative diagram*

$$\begin{array}{ccccc} A' & \xrightarrow{f_1} & A & \xrightarrow{g_1} & A'' \\ \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d''} \\ B' & \xrightarrow{f_2} & B & \xrightarrow{g_2} & B'' \end{array}$$

in $\mathcal{C}$ such that the rows are split exact sequences and that d', d'' are split epimorphisms. Let s', s'' be splits of d', d''. Then there exists $\tilde{s}\colon B \to A$ such that $\tilde{s}f_2 = f_1 s'$, $g_1\tilde{s} = s'' g_2$ and $\gamma := d\tilde{s}$ is an elementary transformation of B. In particular, $s := \tilde{s}\gamma^{-1}$ is a split of d.

Proof. Let a and b are splits of g_1 and f_2 respectively;



We define

$$\tilde{s} := f_1 s' b + a s'' g_2 \colon B \to A.$$

Then $\tilde{s}f_2 = f_1 s'$ and $g_1\tilde{s} = s'' g_2$. Set $\gamma := d\tilde{s}$. Then $\gamma f_2 = f_2$ and $g_2\gamma = g_2$, which implies that γ is an elementary transformation of B. □

2.4. **Euler characteristic II.** In this subsection, we prove that the Euler characteristic defined in Definition 2.6 gives a group homomorphism $\chi\colon K_0(\mathsf{D}(F)) \to K_0(F)$.

Lemma 2.9. *Suppose we are given exact sequences*

$$P' \rightarrowtail P \twoheadrightarrow P'', \quad Q' \rightarrowtail Q \twoheadrightarrow Q''$$

in $\mathsf{Ch}^b(\mathcal{A})$ and homotopy equivalences $\alpha, \alpha', \alpha''$ fitting into a commutative diagram

$$\begin{array}{ccccc} F(P') & \rightarrowtail & F(P) & \twoheadrightarrow & F(P'') \\ \sim\downarrow{\scriptstyle \alpha'} & & \sim\downarrow{\scriptstyle \alpha} & & \sim\downarrow{\scriptstyle \alpha''} \\ F(Q') & \rightarrowtail & F(Q) & \twoheadrightarrow & F(Q''). \end{array}$$

Then

$$\chi(P, \alpha, Q) = \chi(P', \alpha', Q') + \chi(P'', \alpha'', Q'').$$

Proof. We have a commutative diagram

$$\begin{array}{ccccc} Z'_n & \rightarrowtail & Z_n & \twoheadrightarrow & Z''_n \\ \downarrow{\scriptstyle \epsilon'} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon''} \\ F(P'_{n-1}) \oplus F(Q'_n) & \rightarrowtail & F(P_{n-1}) \oplus F(Q_n) & \twoheadrightarrow & F(P''_{n-1}) \oplus F(Q''_n) \\ \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta''} \\ Z'_{n-1} & \rightarrowtail & Z_{n-1} & \twoheadrightarrow & Z''_{n-1} \end{array}$$

with exact rows and columns, where Z_*, Z'_*, Z''_* are the kernels of the differentials of $\operatorname{cone}\alpha, \operatorname{cone}\alpha', \operatorname{cone}\alpha''$. We take splits s' and s'' of δ' and δ''. Let $\tilde{s}$ be a map $Z_{n-1} \to F(P_{n-1}) \oplus F(Q_n)$ as in Lemma 2.8, and set $\gamma := \delta\tilde{s}$, $s := \tilde{s}\gamma$. Then γ is an elementary transformation and s is a split of δ.

Now, we have isomorphisms $\tilde{\phi}_n, \phi'_n, \phi''_n$ fitting into a commutative diagram

$$\begin{array}{ccccc} Z'_n \oplus Z'_{n-1} & \rightarrowtail & Z_n \oplus Z_{n-1} & \twoheadrightarrow & Z''_n \oplus Z''_{n-1} \\ \simeq \downarrow{\scriptstyle \phi'_n = (\epsilon', s')} & & \simeq \downarrow{\scriptstyle \tilde{\phi}_n = (\epsilon, \tilde{s})} & & \simeq \downarrow{\scriptstyle \phi''_n = (\epsilon'', s'')} \\ F(P'_{n-1}) \oplus F(Q'_n) & \rightarrowtail & F(P_{n-1}) \oplus F(Q_n) & \twoheadrightarrow & F(P''_{n-1}) \oplus F(Q''_n). \end{array}$$

Set

$$\Phi' := (\phi'_{2*})^{-1}\phi'_{2*+1}, \quad \Phi'' := (\phi''_{2*})^{-1}\phi''_{2*+1}, \quad \tilde{\Phi} := \tilde{\phi}_{2*}^{-1}\tilde{\phi}_{2*+1}.$$

Then we obtain a sequence in $\mathsf{Rel}(F)$

$$(P'_{2*} \oplus Q'_{2*+1}, \Phi', P'_{2*-1} \oplus Q'_{2*}) \to (P_{2*} \oplus Q_{2*+1}, \tilde{\Phi}, P_{2*-1} \oplus Q_{2*}) \to (P''_{2*} \oplus Q''_{2*+1}, \Phi'', P'_{2*-1} \oplus Q''_{2*}),$$

and it is an exact sequence since the given exact sequences are degree-wise exact.

On the other hand, $\tilde{\Phi}$ is equal to $\Phi := \phi_{2*}^{-1}\phi_{2*+1}$, $\phi_n := (\epsilon, s)$, modulo elementary transformations. Therefore,

$$\begin{aligned} \chi(P, \alpha, Q) &= [(P_{2*} \oplus Q_{2*+1}, \Phi, P_{2*-1} \oplus Q_{2*})] \\ &= [(P_{2*} \oplus Q_{2*+1}, \tilde{\Phi}, P_{2*-1} \oplus Q_{2*})] \\ &= [(P'_{2*} \oplus Q'_{2*+1}, \Phi', P'_{2*-1} \oplus Q'_{2*})] + [(P''_{2*} \oplus Q''_{2*+1}, \Phi'', P'_{2*-1} \oplus Q''_{2*})] \\ &= \chi(P', \alpha', Q') + \chi(P'', \alpha'', Q''). \end{aligned}$$

□

Lemma 2.10. *Let P, Q be bounded complexes in $\mathcal{A}$ with a homotopy equivalence $\alpha\colon F(P) \xrightarrow{\sim} F(Q)$ such that $H_*X, H_*Y \in \mathcal{A}$. Then we have*

$$\chi(P, \alpha, Q) = \sum_i (-1)^i [(H_iP, H_i\alpha, H_iQ)].$$

Proof. Since $\chi(P[1], \alpha[1], Q[1])] = -\chi(P, \alpha, Q)$ by Lemma 2.7 (ii), we may assume that $P_i = Q_i = 0$ for $i < 0$. It is easy to see from our assumptions that the kernels and the images of $d_P\colon P_n \to P_{n-1}$ and $d_Q\colon Q_n \to Q_{n-1}$ are in $\mathcal{A}$. Now, we have an exact sequence

$$(\tau_{\geq n+1}P, \tau_{\geq n+1}\alpha, \tau_{\geq n+1}Q) \rightarrowtail (P, \alpha, Q) \twoheadrightarrow (\tau_{\leq n}P, \tau_{\leq n}\alpha, \tau_{\leq n}Q).$$

By Lemma 2.9 and by induction, we may assume that $P_i = Q_i = 0$ for $i \geq 2$ and that $d_P\colon P_1 \to P_0$ and $d_Q\colon Q_1 \to Q_0$ are admissible monomorphisms. Now, the cone of α has the form

$$F(P_1) \to F(P_0) \oplus F(Q_1) \to F(Q_0),$$

which fits into a commutative diagram

$$\begin{array}{ccccc}
 & & F(P_1) & = & F(P_1) \\
 & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle d} \\
F(Q_1) & \rightarrowtail & F(P_0) \oplus F(Q_1) & \twoheadrightarrow & F(P_0) \\
\| & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \bar{\alpha}} \\
F(Q_1) & \rightarrowtail & F(Q_0) & \twoheadrightarrow & F(H_0(Q))
\end{array}$$

with exact rows and columns. We take a split s'' of $\bar{\alpha}$, and take $\tilde{s}\colon F(Q_0) \to F(P_0) \oplus F(Q_1)$ as in Lemma 2.8. Then we have a commutative diagram

$$\begin{array}{ccccc}
F(Q_1) & \rightarrowtail & F(P_0) \oplus F(Q_1) & \twoheadrightarrow & F(P_0) \\
\| & & \simeq \downarrow{\scriptstyle \tilde{\phi}:=(\epsilon,\tilde{s})} & & \simeq \downarrow{\scriptstyle \psi:=(d,s'')} \\
F(Q_1) & \rightarrowtail & F(Q_0) \oplus F(P_1) & \twoheadrightarrow & F(H_0(Q)) \oplus F(P_1).
\end{array}$$

Since the rows lift canonically to exact sequences in $\mathcal{A}$, we have

$$\chi(P, \alpha, Q) = [(P_0 \oplus Q_1, \tilde{\phi}, P_1 \oplus Q_0)] = [(P_0, \psi, H_0(Q) \oplus P_1)].$$

Finally, it follows from the exact sequence

$$(P_1, 1, P_1) \rightarrowtail (P_0, \psi, H_0(Q) \oplus P_1) \twoheadrightarrow (H_0(P), H_0(\alpha), H_0(Q))$$

that

$$[(P_0, \psi, H_0(Q) \oplus P_1)] = [(H_0(P), H_0(\alpha), H_0(Q))].$$

□

Corollary 2.11.

(i) *Let $f\colon P \to P'$ and $g\colon Q \to Q'$ be quasi-isomorphisms of bounded complexes in $\mathcal{A}$ with homotopy equivalences $\alpha\colon F(P) \xrightarrow{\sim} F(Q)$ and $\alpha'\colon F(P') \xrightarrow{\sim} F(Q')$ such that $\alpha' F(f) = F(g)\alpha$. Then*

$$\chi(P, \alpha, Q) = \chi(P', \alpha', Q').$$

(ii) *Let P, Q be bounded complexes in $\mathcal{A}$ with a homotopy equivalence $\alpha\colon F(P) \xrightarrow{\sim} F(Q)$. Suppose that α is homotopic to another homotopy equivalence $\beta\colon F(P) \xrightarrow{\sim} F(Q)$. Then*

$$\chi(P, \alpha, Q) = \chi(P, \beta, Q).$$

Proof. Let $C(F(P))$ be the mapping cylinder of the identity map of $F(P)$. Then α and β extend to a homotopy equivalence $C(F(P)) \to F(Q)$. Since the canonical map $F(P) \to C(F(P))$ lifts to a quasi-isomorphism in $\mathcal{A}$, (ii) follows from (i).

For (i), we have an exact sequence

$$(P', \alpha', Q') \rightarrowtail (\operatorname{cone} f, \gamma, \operatorname{cone} g) \twoheadrightarrow (P[1], \alpha[1], Q[1]).$$

According to Lemma 2.9, we have

$$\begin{aligned}
\chi(\operatorname{cone} f, \gamma, \operatorname{cone} g) &= \chi(P', \alpha', Q') + \chi(P[1], \alpha[1], Q[1]) \\
&= \chi(P', \alpha', Q') - \chi(P, \alpha, Q).
\end{aligned}$$

Since $H_* \operatorname{cone} f = H_* \operatorname{cone} g = 0$, it follows from Lemma 2.10 that

$$\chi(\operatorname{cone} f, \gamma, \operatorname{cone} g) = \sum (-1)^i [(H_i \operatorname{cone} f, H_i \gamma, H_i \operatorname{cone} g)] = 0.$$

□

Lemma 2.12. *Let P, Q, R be bounded complexes in $\mathcal{A}$ with homotopy equivalences $\alpha\colon F(P) \xrightarrow{\simeq} F(Q)$ and $\beta\colon F(Q) \xrightarrow{\simeq} F(R)$. Then*

$$\chi(P, \beta\alpha, R) = \chi(P, \alpha, Q) + \chi(Q, \beta, R).$$

Proof. From the exact sequence

$$(P, \beta\alpha, R) \rightarrowtail (P \oplus Q, \left(\begin{smallmatrix} 0 & -1 \\ \beta\alpha & 0 \end{smallmatrix}\right), Q \oplus R) \twoheadrightarrow (Q, 1, Q),$$

it follows that

$$\chi(P, \beta\alpha, R) = \chi(P \oplus Q, \left(\begin{smallmatrix} 0 & -1 \\ \beta\alpha & 0 \end{smallmatrix}\right), Q \oplus R).$$

Let β^{-1} be a homotopy inverse of β. Then we have homotopy equivalences

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \sim \begin{pmatrix} 0 & \beta^{-1} \\ \beta & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \beta\alpha & 0 \end{pmatrix} \sim \gamma \begin{pmatrix} 0 & -1 \\ \beta\alpha & 0 \end{pmatrix},$$

where γ is a product of elementary transformations. Hence, by Corollary 2.11 (ii) and Lemma 2.7 (i), we have

$$\begin{aligned} \chi(P \oplus Q, \left(\begin{smallmatrix} 0 & -1 \\ \beta\alpha & 0 \end{smallmatrix}\right), Q \oplus R) &= \chi(P \oplus Q, \alpha \oplus \beta, Q \oplus R) \\ &= \chi(P, \alpha, Q) + \chi(Q, \beta, R). \end{aligned}$$

□

Proposition 2.13. *The Euler characteristic defined in Definition 2.6 gives a group homomorphism*

$$\chi\colon K_0(\mathsf{D}(F)) \to K_0(F).$$

Proof. Let $X = (P, \bar{\alpha}, Q)$ be an object of $\mathsf{Rel}(\mathsf{K}^b(\mathcal{A})/\mathsf{K}^{b,\emptyset}(\mathcal{A}) \xrightarrow{\mathsf{D}(F)} \mathsf{K}^b(\mathcal{B}))$; P, Q are bounded complexes in $\mathcal{A}$ and $\bar{\alpha}$ is the homotopy equivalent class of a homotopy equivalence $\alpha\colon F(P) \xrightarrow{\simeq} F(Q)$. Hence, by Corollary 2.11 (ii), the Euler characteristic of X

$$\chi(X) := \chi(P, \alpha, Q)$$

is well-defined. It remains to show that χ kills the relations (a) (b) of $K_0(\mathsf{D}(F))$ in Definition 2.1.

For the relation (b), let $(P, \bar{\alpha}, Q), (Q, \bar{\beta}, R) \in \mathsf{Rel}(\mathsf{D}(F))$. Then $\bar{\beta}\bar{\alpha}$ is a homotopy equivalent class of $\beta\alpha$. Hence, by Lemma 2.12, we have

$$\chi(P, \bar{\beta}\bar{\alpha}, R) = \chi(P, \bar{\alpha}, Q) + \chi(Q, \bar{\beta}, R).$$

For the relation (a), let

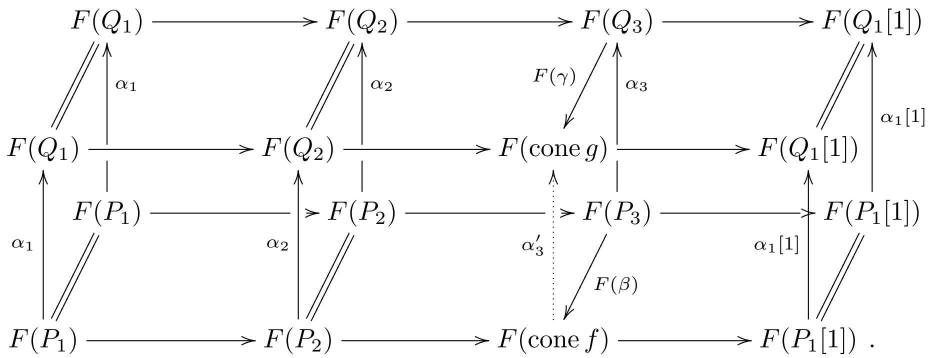
$$(P_1, \alpha_1, Q_1) \xrightarrow{(f,g)} (P_2, \alpha_2, Q_2) \longrightarrow (P_3, \alpha_3, Q_3) \longrightarrow (P_1, \alpha_1, Q_1)[1]$$

be an exact triangle in $\mathsf{Rel}(\mathsf{D}(F))$. According to Corollary 2.11 (i), we may assume that f, g are maps of complexes.

Now, there are isomorphism $\beta\colon P_3 \xrightarrow{\simeq} \operatorname{cone} f$ and $\gamma\colon Q_3 \xrightarrow{\simeq} \operatorname{cone} g$ in $\mathsf{K}^b(\mathcal{A})/\mathsf{K}^{b,\emptyset}(\mathcal{A})$ which make the diagrams

$$\begin{array}{ccccc} P_2 & \longrightarrow & P_3 & \longrightarrow & P_1[1] \\ \| & & \downarrow{\scriptstyle \beta} & & \| \\ P_2 & \longrightarrow & \operatorname{cone} f & \longrightarrow & P_1[1], \end{array} \qquad \begin{array}{ccccc} Q_2 & \longrightarrow & Q_3 & \longrightarrow & Q_1[1] \\ \| & & \downarrow{\scriptstyle \beta} & & \| \\ Q_2 & \longrightarrow & \operatorname{cone} g & \longrightarrow & Q_1[1], \end{array}$$

commutative. We have a homotopy equivalence $\alpha_3' \colon F(\operatorname{cone} f) \xrightarrow{\sim} F(\operatorname{cone} g)$ fitting into the commutative diagram



By Corollary 2.11 (i), we have

$$\chi(P_3, \alpha_3, Q_3) = \chi(\operatorname{cone} f, \alpha_3', \operatorname{cone} g).$$

By Lemma 2.9, we conclude that

$$\begin{aligned}\chi(\operatorname{cone} f, \alpha_3', \operatorname{cone} g) &= \chi(P_2, \alpha_2, Q_2) + \chi(P_1[1], \alpha_1[1], Q_1[1]) \\ &= \chi(P_2, \alpha_2, Q_2) - \chi(P_1, \alpha_1, Q_1).\end{aligned}$$

□

2.5. **Proof of Theorem 2.4.** We prove that the Euler characteristic $\chi \colon K_0(\mathsf{D}(F)) \to K_0(F)$ is the inverse of the canonical map $\iota \colon K_0(F) \to K_0(\mathsf{D}(F))$. It is clear that, for $X \in \mathsf{Rel}(F)$, $\chi\iota[X] = [X]$. Hence, it remains to prove the following.

Lemma 2.14. *For* $(P, \alpha, Q) \in \mathsf{Rel}(\mathsf{D}(F))$,

$$[(P, \alpha, Q)] = \iota\chi(P, \alpha, Q)$$

in $K_0(\mathsf{D}(F))$.

Proof. We may assume that $P_i = Q_i = 0$ for $i < 0$. We prove the lemma by induction on $N := \min\{n \mid P_i = Q_i = 0 \quad \forall i > n \geq 0\}$. The case $N = 0$ is clear.

We use the following notation: Set

$$\Omega_1 := \bigoplus_i (P_{2i} \oplus Q_{2i+1}) \quad \text{and} \quad \Omega_2 := \bigoplus_i (Q_{2i} \oplus P_{2i+1}),$$

so that $\chi(P, \alpha, Q) = [(\Omega_1, \Phi, \Omega_2)]$. According to Freyd-Mitchell theorem, $\mathcal{A}$ has an embedding into an abelian category of modules, which allows us talking about elements of objects in $\mathcal{A}$. In principle, we shall denote elements of P_i (resp. Q_i) by x_i (resp. y_i).

First of all, we construct $(P', \alpha', Q') \in \mathsf{Rel}(\mathsf{D}(F))$ with morphisms

$$(\Omega_1, \Phi, \Omega_2) \xrightarrow{\theta} (P', \alpha', Q') \xleftarrow{\simeq} (P, \alpha, Q).$$

Here, $Q' := Q$ and

$$P' := P \oplus [\cdots \longrightarrow 0 \longrightarrow Q_1 \xrightarrow{1} Q_1].$$

The quasi-isomorphism $\alpha' \colon F(P') \to F(Q)$ is given by

$$\begin{array}{ccccccc} \cdots \longrightarrow & F(P_2) & \longrightarrow & F(P_1) \oplus F(Q_1) & \longrightarrow & F(P_0) \oplus F(Q_1) \\ & \big\downarrow{\scriptstyle \alpha_2} & & \big\downarrow{\scriptstyle \alpha_1 \oplus 1} & & \big\downarrow{\scriptstyle \alpha_0 \oplus d} \\ \cdots \longrightarrow & F(Q_2) & \longrightarrow & F(Q_1) & \longrightarrow & F(Q_0). \end{array}$$

The canonical inclusion $P \to P'$ is a quasi-isomorphism, and it yields an isomorphism $(P, \alpha, Q) \xrightarrow{\simeq} (P', \alpha', Q)$ in $\mathsf{Rel}(\mathsf{D}(F))$. The map θ is given by

$$\begin{aligned} \Omega_1 &= P_0 \oplus Q_1 \oplus P_2 \oplus \cdots \to P_0' = P_0 \oplus Q_1 && (x_0, y_1, x_2, \dots) \mapsto (-x_0, y_1) \\ \Omega_2 &= Q_0 \oplus P_1 \oplus Q_2 \oplus \cdots \to Q_0 && (y_0, x_1, y_2, \dots) \mapsto y_0. \end{aligned}$$

We show that $[\operatorname{cone}\theta] = 0$ in $K_0(\mathsf{D}(F))$, which proves the lemma. First few degrees of $\operatorname{cone}\theta$ look like

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ P_2 & & Q_2 \\ \downarrow & & \downarrow \\ (P_0 \oplus Q_1 \oplus P_2 \oplus \cdots) \oplus P_1 \oplus Q_1 & & (Q_0 \oplus P_1 \oplus Q_2 \oplus \cdots) \oplus Q_1 \\ \downarrow & & \downarrow \\ P_0 \oplus Q_1 & & Q_0. \end{array}$$

It follows that the class of $\operatorname{cone}\theta$ in $K_0(\mathsf{D}(F))$ is equal to $-[(R, \beta, S)]$ where

$$R = [\cdots \to P_3 \xrightarrow{d_P} P_2 \xrightarrow{0 \oplus d_P} (Q_1 \oplus P_2 \oplus Q_3 \oplus \cdots) \oplus P_1]$$
$$S = [\cdots \to Q_3 \xrightarrow{d_Q} Q_2 \xrightarrow{0 \oplus d_Q} (P_1 \oplus Q_2 \oplus P_3 \oplus \cdots) \oplus Q_1],$$

$\beta_i = \alpha_{i+1}$ for $i \geq 1$ and β_1 is given by

$$\beta_1((y_1, x_2, y_3, \dots,), x_1) = (\mathrm{pr}_2\Phi(-dx_1, y_1, x_2, y_3, \dots), \alpha x_1 + y_1).$$

The induction hypothesis implies that $[(R, \beta, S)] = \iota\chi(R, \beta, S)$. We show that $\chi(R, \beta, S) = 0$.

We write

$$\Omega_1' := \bigoplus_{i \geq 1}(Q_{2i-1} \oplus P_{2i}) \quad \text{and} \quad \Omega_2' := \bigoplus_{i \geq 1}(P_{2i-1} \oplus Q_{2i}),$$

and denote the projections $\Omega_l \to \Omega_l'$ by pr_l. Then we have

$$\chi(R, \beta, S) = [(\Omega_1' \oplus \Omega_2', \Phi', \Omega_2' \oplus \Omega_1')],$$

where Φ' is given by

$$((y_1, x_2, y_3, \dots), (x_1, y_2, x_3, \dots)) \mapsto (-\mathrm{pr}_2\Phi(-dx_1, y_1, x_2, y_3, \dots), -y_1 + \mathrm{pr}_1\Phi^{-1}(0, x_1, y_2, x_3, \dots)).$$

Observe that we have $\chi(R, \beta, S) \looparrowright (\Omega_1 \oplus \Omega_2, \Psi, \Omega_2 \oplus \Omega_1)$ (see Definition 1.7 for "$\looparrowright$") with

$$\begin{aligned} \Psi \colon &((x_0, y_1, x_2, \dots), (y_0, x_1, y_2, \dots)) \\ &\mapsto ((y_0, -\mathrm{pr}_2\Phi(-dx_1 + x_0, y_1, x_2, y_3, \dots)), (-x_0, -y_1 + \mathrm{pr}_1\Phi^{-1}(y_0, x_1, y_2, x_3, \dots))). \end{aligned}$$

Since the class of $(\Omega_2 \oplus \Omega_1, -\Phi^{-1} \oplus \Phi, \Omega_2 \oplus \Omega_1)$ is zero, we have

$$\chi(R, \beta, S) = [(\Omega_2 \oplus \Omega_1, \Psi(-\Phi^{-1} \oplus \Phi), \Omega_2 \oplus \Omega_1)].$$

Now, $\Psi(-\Phi^{-1} \oplus \Phi)$ is given by

$$\begin{aligned} &((y_0, x_1, y_2, x_3 \dots), (x_0, y_1, x_2, y_3 \dots)) \\ \mapsto &((-\alpha x_0 + dy_1, x_1 + A, y_2 + B, x_3, y_4, \dots), (s(y_0)_{P_0} - dx_1, y_1 - \alpha x_1 + dy_2 + s(y_0)_{Q_1}, x_2, y_3, \dots)), \end{aligned}$$

where $A := s((d_P s(x_0, y_1)_{P_1}, 0))_{P_1}$, $B := s((d_P s(x_0, y_1)_{P_1}, 0))_{Q_2}$ and s is the split $Q_0 \to P_0 \oplus Q_1$ or $P_0 \oplus Q_1 \to P_1 \oplus Q_2$. Hence,

$$\begin{aligned} &((Q_0 \oplus P_1 \oplus Q_2) \oplus (P_0 \oplus Q_1), \Psi', (Q_0 \oplus P_1 \oplus Q_2) \oplus (P_0 \oplus Q_1)) \\ &\qquad\qquad \looparrowright (\Omega_2 \oplus \Omega_1, \Psi(-\Phi^{-1} \oplus \Phi), \Omega_2 \oplus \Omega_1), \end{aligned}$$

where Ψ' is the restriction of $\Psi(-\Phi^{-1} \oplus \Phi)$. We calculate the class of the left hand side in $K_0(F)$ and show that it is zero.

We set $M_0 := Q_0$, $M_2 := P_1 \oplus Q_2$, $M_1 := P_0 \oplus Q_1$, and denote by $\delta\colon M_l \to M_{l-1}$ the differential of the cone of α. Since $-d_P s(x_0, y_1)_{P_1} + (s\delta(x_0, y_1))_{P_0} = x_0$, we have

$$(A, B) = -s(x_0) + s(s(\delta(x_0, y_1))_{P_0}).$$

Let p and q be the projections $M_1 \to P_0$ and $M_1 \to Q_1$ respectively. Then Ψ' is expressed by the matrix (an endomorphism of $F(M_0) \oplus F(M_2) \oplus F(M_1)$)

$$\Psi' = \begin{pmatrix} 0 & 0 & \delta \\ 0 & 1 & -sp + sps\delta \\ s & \delta & p_{Q_1} \end{pmatrix}$$

and we have

$$\begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -sps & 1 & 0 \\ 0 & -\delta & 1 \end{pmatrix} \Psi' = \begin{pmatrix} -1 & 0 & \delta p \\ 0 & 1 & -sp \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, Ψ' lifts to an automorphism of $M_0 \oplus M_2 \oplus M_1$ modulo elementary transformations, and thus

$$[(M_0 \oplus M_2 \oplus M_1, \Psi', M_0 \oplus M_2 \oplus M_1)] = 0.$$

□

3. Relative cycle class map

3.1. **Relative K-theory of schemes.** For a scheme X, we use the following notation:

(1) $\mathsf{Vec}(X)$ is the category of algebraic vector bundles on X.
(2) $K(X)$ is Quillen's K-theory spectrum of the exact category $\mathsf{Vec}(X)$.
(3) $\mathsf{D}^b(X)$ is the derived category of bounded complexes of $\mathcal{O}_X$-modules.
(4) $\mathsf{D}^{\mathrm{perf}}(X) \subset \mathsf{D}^b(X)$ is the full subcategory of perfect complexes.

For the most part in this section, we shall assume that a scheme has an ample family of line bundles, cf. [TT90, 2.1.1]. For example, any scheme quasi-projective over an affine scheme has an ample family of line bundles. Also, any separated regular noetherian scheme has an ample family of line bundles [SGA6, II 2.2.7.1]. If a scheme X has an ample family of line bundles, then Quillen's K-theory spectrum $K(X)$ behaves well, that is, it is equivalent to the K-theory spectrum of the Waldhausen category of perfect complexes of X, cf. [TT90, 3.9].

The following theorem is a consequence of the results in §1 and §2.

Theorem 3.1. *Let X be a scheme with an ample family of line bundles, Y an affine scheme and $f\colon Y \to X$ a morphism of schemes. Then there exists a natural isomorphism*

$$\pi_0 \operatorname{hofib}(K(X) \xrightarrow{f^*} K(Y)) \simeq K_0(\mathsf{D}^{\mathrm{perf}}(X) \xrightarrow{Lf^*} \mathsf{D}^{\mathrm{perf}}(Y)).$$

See Definition 2.1 for the definition of the right group.

Proof. Since X has an ample family of line bundles, every perfect complex is quasi-isomorphic to a bounded complex of algebraic vector bundles, and thus there is an equivalence of triangulated categories

$$\mathsf{K}^b(\mathsf{Vec}(X))/\mathsf{K}^{b,\emptyset}(\mathsf{Vec}(X)) \xrightarrow{\simeq} \mathsf{D}^{\mathrm{perf}}(X).$$

Since Y is affine, $\mathsf{Vec}(Y)$ is split exact and $\mathsf{D}^{\mathrm{perf}}(Y) \simeq \mathsf{K}^b(\mathsf{Vec}(Y))$. Now, the triangulated functor $Lf^*\colon \mathsf{D}^{\mathrm{perf}}(X) \to \mathsf{D}^{\mathrm{perf}}(Y)$ is identified with the functor $D(f^*)$ induced from the exact functor $f^*\colon \mathsf{Vec}(X) \to \mathsf{Vec}(Y)$, cf. §2.2. Therefore, by Theorem 2.4, we have an isomorphism

$$K_0(\mathsf{Vec}(X) \xrightarrow{f^*} \mathsf{Vec}(Y)) \simeq K_0(\mathsf{D}^{\mathrm{perf}}(X) \xrightarrow{Lf^*} \mathsf{D}^{\mathrm{perf}}(Y)).$$

By Theorem 1.5, the left hand side is isomorphic to $\pi_0 \operatorname{hofib}(K(X) \xrightarrow{f^*} K(Y))$, and we get the theorem. □

Suppose we are given two schemes X, Y and a morphism of schemes $f\colon Y \to X$ between them. If the morphism f is obvious from the context, we denote by $K(X, Y)$ the homotopy fiber of $f^*\colon K(X) \to K(Y)$ and write $K_0(X, Y) := \pi_0 K(X, Y)$. We adapted this notation because our main interest is the case Y is a closed subscheme of X; in this case, the map $f\colon Y \to X$ is the canonical inclusion.

It is clear from the definition that $K_0(X, Y)$ is contravariant functorial, i.e. a commutative diagram

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

induces a group homomorphism $K_0(X', Y') \to K_0(X, Y)$.

According to the base change theorem [SGA6, IV 3.1.1][1], we have a proper transfer of $K_0(X, Y)$ in the following case.

Proposition 3.2. *Suppose we are given a cartesian diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

of schemes. Assume that:

(i) X, X' *have ample family of line bundles and* Y, Y' *are affine.*
(ii) f *and* g *are Tor-independent over* X *[loc. cit., III 1.5].*
(iii) g *is proper and perfect [loc. cit., III 4.1].*

Then there is a map

$$K_0(X', Y') \to K_0(X, Y)$$

which sends $(P, \alpha, Q) \in \mathsf{Rel}(Lf'^*)$ *to* $(Rg_*P, Rg'_*\alpha, Rg_*Q) \in \mathsf{Rel}(Lf^*)$.

3.2. **Coniveau filtration.** Let X be a scheme of dimension d which has an ample family of line bundles, Y an affine closed subscheme of X, and we denote the inclusion $Y \hookrightarrow X$ by ι. We assume that $X \setminus Y$ is regular. This is a standing assumption that prevails in §3.2.

By Theorem 3.1, we identify $K_0(X, Y)$ with the K_0 of the triangulated functor $Lf^*\colon \mathsf{D}^{\mathrm{perf}}(X) \to \mathsf{D}^{\mathrm{perf}}(Y)$.

Definition 3.3.

(i) For $\mathfrak{A} = (P, \alpha, Q) \in \mathsf{Rel}(L\iota^*)$, let $S_{\mathfrak{A}}$ be the set of open neighborhoods U of Y in X such that there exits an isomorphism $\tilde{\alpha}\colon P|_U \xrightarrow{\simeq} Q|_U$ in $\mathsf{D}^{\mathrm{perf}}(U)$ which lifts α.
(ii) For $-1 \le i \le d$, we define $F_i K_0(X, Y)$ to be the subgroup of $K_0(X, Y)$ generated by elements $\mathfrak{A} \in \mathsf{Rel}(L\iota^*)$ for which there exists $U \in S_{\mathfrak{A}}$ with $\dim(X \setminus U) \le i$.

By the definition, $F_i K_0(X, Y) \subset F_{i+1} K_0(X, Y)$, and $F_{-1} K_0(X, Y)$ is generated by (P, α, Q) for which there exists $\tilde{\alpha}\colon P \xrightarrow{\simeq} Q$ such that $L\iota^*\tilde{\alpha} = \alpha$. Hence, it follows from Lemma 2.2 that $F_{-1} K_0(X, Y) = 0$. In general, $F_d K_0(X, Y)$ may not be equal to $K_0(X, Y)$. However, we have:

Lemma 3.4. *If* Y *has an affine open neighborhood in* X*, then* $F_d K_0(X, Y) = K_0(X, Y)$.

Proof. Let $\mathfrak{A} \in \mathsf{Rel}(L\iota^*)$. According to Theorem 2.4, the class of $\mathfrak{A}$ in $K_0(X, Y)$ is equal to the one of some (P, α, Q) with $P, Q \in \mathsf{Vec}(X)$. It suffices to show that $S_{(P,\alpha,Q)} \neq \emptyset$, i.e. there exists an open neighborhood U of Y in X and an isomorphism $P|_U \xrightarrow{\simeq} Q|_U$ which lifts α.

[1]"La conjecture de finitude [loc. cit., III 2.1]" assumed there has been proved by Kiehl in [Ki72].

By our assumption, we may assume that X is affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/I$. Since Q is a projective A-module, we have an A-homomorphism $\gamma\colon P \to Q$ which fits into the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\gamma} & Q \\ \downarrow & & \downarrow \\ P \otimes_A A/I & \xrightarrow{\alpha} & Q \otimes_A A/I. \end{array}$$

Let K and L be the kernel and the cokernel of γ respectively. We claim that for every $y \in Y$, $K_y = L_y = 0$. Since $- \otimes_{A_y} A_y/\mathfrak{m}_y$ ($\mathfrak{m}_y$ is the maximal ideal) is right exact and $\gamma_y \otimes_{A_y} A_y/\mathfrak{m}_y$ is an isomorphism, we have $L_y \otimes_{A_y} A_y/\mathfrak{m}_y = 0$. By Nakayama's lemma, $L_y = 0$. Since Q_y is projective, the exact sequence

$$0 \longrightarrow K_y \longrightarrow P_y \longrightarrow Q_y \longrightarrow 0$$

is split exact. Hence, $K_y \otimes_{A_y} A_y/\mathfrak{m}_y = 0$ and $K_y = 0$.

Since the supports of K and L are closed, it follows from the claim that there exists some open neighborhood of Y on which γ is an isomorphism. This completes the proof. □

Let $C_k(X|Y)$ be the set of integral closed subschemes of X of dimension k which do not meet Y, and $Z_k(X|Y)$ the free abelian group generated by $C_k(X|Y)$. For $V \in C_k(X|Y)$, the triple $(\mathcal{O}_V, 0, 0)$ defines an element of $F_kK_0(X, Y)$, which we denote by $\mathsf{cyc}(V)$.

Lemma 3.5. *Under the standing assumption in §3.2, the map*

$$\mathsf{cyc}\colon Z_k(X|Y) \to F_kK_0(X, Y)/F_{k-1}K_0(X, Y)$$

is surjective for all $k \geq 0$.

Proof. Suppose we are given $\mathfrak{A} \in \mathsf{Rel}(L\iota^*)$ whose class is in $F_kK_0(X, Y)$ but not in F_{k-1}, so that there exists $U \in S_{\mathfrak{A}}$ such that $\dim(X \setminus U) = k$. Set $V := X \setminus U$ equipped with the reduced scheme structure.

Let $K^V(X)$ be the K-theory spectrum of X with support in V, i.e. the homotopy fiber of the canonical map $K(X) \to K(U)$. Since V does not meet Y, $K^V(X)$ is equivalent to the homotopy fiber of $K(X, Y) \to K(U, Y)$. Also, since V does not meet Y and $X \setminus Y$ is regular, $K^V(X)$ is equivalent to the G-theory spectrum $G(V)$ of V. Hence, we have an exact sequence

$$G_0(V) \longrightarrow K_0(X, Y) \longrightarrow K_0(U, Y).$$

Now, the class of $\mathfrak{A}$ dies in $K_0(U, Y)$, and thus it comes from $G_0(V)$.

Since the usual cycle map

$$\bigoplus_{i=0}^{k} Z_i(V) \to G_0(V), \quad W \mapsto \mathcal{O}_W$$

is surjective, the class of $\mathfrak{A}$ is in the image of

$$\bigoplus_{i=0}^{k} Z_i(V) \to \bigoplus_{i=0}^{k} Z_i(X|Y) \to F_kK_0(X, Y).$$

This proves the lemma. □

3.3. **Chow group with modulus.** Let X be a scheme separated of finite type over a field k and D an effective Cartier divisor on X. We denote the inclusion $D \hookrightarrow X$ by ι. We recall the definition of the Chow group with modulus by Binda-Saito [BS17].

Let $k \geq 0$. Let $R_k(X|D)$ be the set of integral closed subschemes V of $X \times \mathbb{P}^1$ of dimension $k+1$ which are dominant over $\mathbb{P}^1$ and satisfy the following condition (modulus condition): Let V^N be the normalization of V and ϕ the canonical map $V^N \to X \times \mathbb{P}^1$, then we have an inequality of Cartier divisors

$$\phi^*(D \times \mathbb{P}^1) \leq \phi^*(X \times \{\infty\}).$$

For each $V \in R_k(X|D)$, the inverse images V_t of $t \in \mathbb{P}^1$ are purely of dimension k and do not meet D, and thus define elements of $Z_k(X|D)$ in the standard way.

Definition 3.6 (Binda-Saito [BS17])**.** The *Chow group with modulus* $\mathrm{CH}_k(X|D)$ is defined to be the quotient of $Z_k(X|D)$ by the relations $[V_0] = [V_1]$ for all $V \in R_k(X|D)$.

Theorem 3.7. *Suppose that X is regular and that D is affine. Then there is a surjective group homomorphism*

$$\mathsf{cyc}\colon \mathrm{CH}_k(X|D) \twoheadrightarrow F_k K_0(X,D)/F_{k-1}K_0(X,D)$$

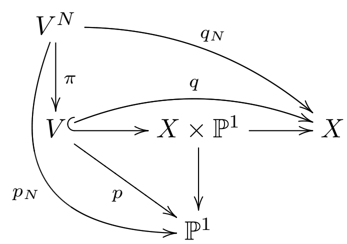
for every $k \geq 0$.

Proof. Since X is separated regular noetherian, it has an ample family of line bundles [SGA6, II 2.2.7.1]. Hence, the assumption in §3.2 is satisfied, and by Lemma 3.5, we have a surjective homomorphism

$$\mathsf{cyc}\colon Z_k(X|D) \twoheadrightarrow F_k K_0(X,D)/F_{k-1}K_0(X,D).$$

We show this map factors through $\mathrm{CH}_k(X|D)$.

We have to show that, for all $V \in R_k(X|D)$, $\mathsf{cyc}(V_0) = \mathsf{cyc}(V_1)$ in $F_k K_0(X,D)/F_{k-1}K_0(X,D)$. Let $V \in R_k(X|D)$ and consider the diagram



where π is the normalization and the other maps are the obvious ones. We fix a parameter t of $\mathbb{P}^1 \setminus \{\infty\}$. Let $j_0\colon \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^1}$ be the canonical inclusion (sending t to t) and $j_1\colon \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^1}$ the map sending t to $t-1$. Now, V_t ($t = 0, 1$) are the closed subschemes of V defined by $p^* j_t$, and these are regarded as closed subschemes of X by the restriction of q.

By Theorem 3.1, we identify $K_0(V^N, q_N^* D)$ with the K_0 of the triangulated functor $L\iota'^*\colon \mathsf{D}^{\mathrm{perf}}(V^N) \to \mathsf{D}^{\mathrm{perf}}(q_N^* D)$, where $\iota'\colon q_N^* D \hookrightarrow V^N$ is the canonical inclusion. Let V_t' be the scheme theoretic inverse image of V_t by $\pi\colon V^N \to V$. Then the triple $(\mathcal{O}_{V_t'}, 0, 0)$ gives an element of $\mathsf{Rel}(L\iota'^*)$.

First, we show that

$$[(\mathcal{O}_{V_0'}, 0, 0)] = [(\mathcal{O}_{V_1'}, 0, 0)]$$

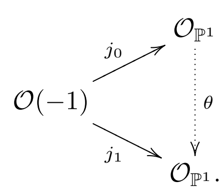
in $K_0(V^N, q_N^* D)$. Now, we have an exact sequence

$$0 \longrightarrow p_N^* \mathcal{O}(-1) \xrightarrow{p_N^* j_t} \mathcal{O}_{V^N} \longrightarrow \mathcal{O}_{V_t'} \longrightarrow 0$$

of $\mathcal{O}_{V^N}$-modules. Hence, the class of $(\mathcal{O}_{V_t'}, 0, 0)$ in $K_0(V^N, q^* D)$ is equal to the one of

$$(p_N^* \mathcal{O}(-1), p_N^* j_t, \mathcal{O}_{V^N}).$$

Let θ be the multiplication of $(t-1)/t$, which is defined on $\mathbb{P}^1 \setminus \{0\}$ and an automorphism on $\mathbb{P}^1 \setminus \{0, 1\}$; θ fits into the commutative diagram



It follows that

$$[(p_N^* \mathcal{O}(-1), p_N^* j_0, \mathcal{O}_{V^N})] + [(\mathcal{O}_{V^N}, p_N^* \theta, \mathcal{O}_{V^N})] = [(p_N^* \mathcal{O}(-1), p_N^* j_1, \mathcal{O}_{V^N})].$$

It is clear that the restriction of $p_N^*\theta$ on $q_N^*(\{\infty\}) = \phi^*(X \times \{\infty\})$ is the identity. Hence, by the modulus condition, the restriction of $p_N^*\theta$ on $\phi^*(D \times \mathbb{P}^1) = q_N^* D$ is the identity. This implies that the second term of the above equation is zero; in other words,

$$[(p_N^*\mathcal{O}(-1), p_N^* j_0, \mathcal{O}_{V^N})] = [(p_N^*\mathcal{O}(-1), p_N^* j_1, \mathcal{O}_{V^N})].$$

This proves $[(\mathcal{O}_{V_0'}, 0, 0)] = [(\mathcal{O}_{V_1'}, 0, 0)]$ in $K_0(V^N, q_N^* D)$.

Now, $\iota\colon D \hookrightarrow X$ and $q_N\colon V^N \to X$ are Tor-independent, and q_N is proper and perfect since X is regular. Hence, by Proposition 3.2, we have a transfer map

$$q_{N*}\colon K_0(V^N, q_N^* D) \to K_0(X, D).$$

Consequently, we have

$$[(q_{N*}\mathcal{O}_{V_0'}, 0, 0)] = [(q_{N*}\mathcal{O}_{V_1'}, 0, 0)]$$

in $K_0(X, D)$.

Finally, we claim that

$$\mathsf{cyc}(V_t) \equiv [(\mathcal{O}_{V_t}, 0, 0)] \equiv [(q_{N*}\mathcal{O}_{V_t'}, 0, 0)]$$

modulo $F_{k-1}K_0(X, D)$, which completes the proof. The first term is $\sum_i m_i[(\mathcal{O}_{V_{t,i}}, 0, 0)]$ by definition, where $V_{t,i}$ are irreducible components of V_t and m_i are their multiplicity. By Lemma 3.8 below, it suffices to compare the length of $\mathcal{O}_{V_t}$ and $q_{N*}\mathcal{O}_{V_t'}$ at the generic point of $V_{t,i}$. This is clear because $V_t \hookrightarrow V$ and $V_t' \hookrightarrow V^N$ are defined by the same rational function. □

Lemma 3.8. *Let $\mathcal{F}$ be a coherent sheaf on X whose support is of dimension k and disjoint from D. Then*

$$[(\mathcal{F}, 0, 0)] = \sum_{\dim V = k} m_V(\mathcal{F})[(\mathcal{O}_V, 0, 0)]$$

in $F_k K_0(X, D)/F_{k-1}K_0(X, D)$. Here, V runs over all integral closed subschemes of X of dimension k and $m_V(\mathcal{F})$ is the length of the stalk of $\mathcal{F}$ at the generic point of V.

Proof. Let $j\colon Z \hookrightarrow X$ be the scheme theoretic support of $\mathcal{F}$, i.e. $\mathcal{F} = j_*\mathcal{G}$ for some coherent module $\mathcal{G}$ of Z. The map $G_0(Z) \to K_0(X, D)$ sending $[\mathcal{G}]$ to $[(j_*\mathcal{G}, 0, 0)]$ is compatible with the coniveau filtration. Hence, it suffices to show that

$$[\mathcal{G}] = \sum_{\dim V = k} m_V(\mathcal{G})[\mathcal{O}_V]$$

in $F_k G_0(Z)/F_{k-1}G_0(Z)$. This is easily verified by induction. □

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