

LEARNING SEMINAR ON THE FARGUES–FONTAINE CURVE

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OUTLINE

Talk 0 Introduction (Emanuel)

Talk 1 Adic spaces and the equal characteristic Fargues–Fontaine curve

Give a reminder on adic spaces following the presentation in [Sch21, Lecture 2] and [SW20, Lectures 2–4]. Begin by explaining in rough terms the idea behind rigid-analytic spaces [SW20, p. 8]. Define Huber rings, Tate rings [SW20, § 2.2] and the adic spectrum of a Huber pair [SW20, § 2.3]. As an example, explain the Tate algebra and adic unit disk \mathbf{B}_C of an algebraically closed nonarchimedean field C [SW20, Ex. 2.3.7]. Discuss rational open subsets [SW20, § 3.1] and adic spaces [SW20, § 3.2] and state the equivalence between rigid-analytic spaces and adic spaces locally of finite type over C [Sch21, § 3.2]. As an example, mention the punctured open unit disk \mathbf{D}_C^* [SW20, p. 28]. Finally, motivate and introduce the equal characteristic Fargues–Fontaine curve [Sch21, § 2.2, p. 11] as well as its classical points.

Talk 2 Perfectoid spaces

Give a reminder on perfectoid spaces and tilting, following the presentation in [SW20, Lectures 6–7]. Define perfectoid fields, rings [SW20, § 6.1] and spaces [SW20, § 7.1]. Present some examples as in [SW20, Ex. 6.1.5] and $\mathbf{Q}_p(\overline{p^{1/p^\infty}})$. Define the tilt of a perfectoid Tate ring [SW20, Def. 6.2.1] and explain tilting for $\mathbf{Q}_p(\overline{p^{1/p^\infty}})$, $\mathbf{Q}_p^{\text{cycl}}$, $\mathbf{F}_p((t^{1/p^\infty}))$. Then state the tilting equivalence from [SW20, Thm. 7.1.1] and [Sch21, § 3.5.3]; if you like, give some intuition for the proof, but do not spend too much time on it. Discuss the étale site [SW20, § 7.5] and the proétale site [SW20, § 8.2] [Sch21, § 5.4]. If time permits, mention that in order to get a notion which is local on the base, one should consider quasiproétale morphisms and base changes to totally disconnected perfectoid spaces [SW20, § 9.1–9.2] [Sch21, Def. 6.7].

Talk 3 Witt vectors and the mixed characteristic Fargues–Fontaine curve

Introduce (p -typical, ramified) Witt vectors and Teichmüller lifts as in [Ans20, Lecture 3]. Recall tilting and present the approach via Witt vectors (see also [SW20, Thm. 6.2.7, Thm. 6.2.11]). Define $A_{\text{inf}} = W_{\mathcal{O}_E}(\mathcal{O}_C)$ and describe $\text{Spec}(A_{\text{inf}})$ and $\text{Spa}(A_{\text{inf}}, A_{\text{inf}})$, following [Ans20, p. 22] and [SW20, § 12.2]. Now introduce the mixed characteristic Fargues–Fontaine curve, following [Sch21, § 3]. Motivate and give the definitions of $Y_{C,E}$ and $X_{C,E}$. Then state [Sch21, Thm. 3.3] and explain its proof (see also [FS21, § II.1.1] for details, specializing to $S = \text{Spa}(C, \mathcal{O}_C)$).

Talk 4 Isocrystals and their associated vector bundles

Discuss isocrystals and their connections to the Fargues–Fontaine curve, following [Sch21, Lecture 4] and [Ans20, Lecture 11]. Define isocrystals [Ans20, Def. 11.6] and give some examples [Sch21, Ex. 4.1]. Then explain the general Harder–Narasimhan slope formalism [FF18, § 5.5.1]. Quickly discuss the example of vector bundles on curves [FF18, § 5.5.2.1], then the more pertinent example of isocrystals [FF18, § 5.5.2.3], [Ans20, p. 62–63]. Then state the Dieudonné–Manin classification and sketch its proof [Sch21, Thm. 4.2] [Ans20, Thm. 11.7]. Now define the functor from isocrystals

to vector bundles on the Fargues–Fontaine curve. State that $\mathrm{Pic}(X_{C,E}) \simeq \mathbf{Z}$ (we will prove this later) and deduce a Harder–Narasimhan formalism for vector bundles on the Fargues–Fontaine curve. State the classification of vector bundles [Sch21, Thm. 4.4]. Mention that the functor is not fully faithful and briefly mention [Sch21, Ex. 4.6, Ex. 4.7]. The proof will be the content of the next three talks. If time permits, cover the first paragraph of the next talk.

Talk 5 Diamonds

Explain the key point and idea of the proof of the classification result from last time on [Sch21, p. 20]: we need to show that $H^0(X_{C,E}, \mathcal{E}) \neq 0$ for semistable vector bundles \mathcal{E} of slope 0. For this, we will consider $X_{S,E}$ for all perfectoid rings and show that the resulting functor $H^0(X_{-,E}, \mathcal{E})$ is representable in a certain sense. With this motivation, introduce the relative Fargues–Fontaine curves as an analytic adic space [Sch21, § 5.2] [FS21, Prop. II.1.1]. Mention that there is no map $X_S \rightarrow S$ (but still a morphism on étale sites).

To show what the functors $S \mapsto H^0(X_{S,E}, \mathcal{E})$ might look like, give [Sch21, Ex. 4.15, Ex. 4.16]. So we want to consider quotients of perfectoid spaces by proétale equivalence relations. Introduce diamonds, following [Sch21, Lecture 5–6] and [SW20, Lectures 8–9]. If necessary and time permits, give a reminder on the (big) proétale site for perfectoid spaces from Talk 2 [SW20, § 8.2] [Sch21, § 5.4]. If it has not been done in Talk 2, mention that in order to get a notion which is local on the base, one should consider quasiproétale morphisms and base changes to totally disconnected perfectoid spaces [SW20, § 9.1–9.2] [Sch21, Def. 6.7]. Now introduce diamonds [Sch21, Def. 6.6] [SW20, § 8.3]. As an example, discuss $\mathrm{Spd} \mathbf{Q}_p$ [SW20, § 8.4, 9.4]. Define the underlying topological space of a diamond and spatial diamonds [Sch21, Def. 6.9] [SW20, § 10.3] and state the embedding from analytic adic spaces over \mathbf{Z}_p into diamonds [Sch21, § 6.4] [SW20, § 10.1]. Finish by explaining the diamond formula for relative Fargues–Fontaine curves [Sch21, § 6.5] [FS21, Prop. II.1.2, II.1.17] (cf. the structure of $\mathrm{Spd} \mathbf{Q}_p \times \mathrm{Spd} \mathbf{Q}_p$ [SW20, Ex. 10.1.8]).

Talk 6 Link with p-divisible groups

The goal of this talk is to compute $H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1))$ from Lubin–Tate theory, following [Sch21, § 7]. Define closed Cartier divisors and explain how untilts of S give rise to such [Sch21, § 7.2] [FS21, Prop. II.1.4, Prop. II.1.18, Def. II.1.19] (see also [SW20, § 5.3]). Discuss Lubin–Tate formal groups [Sch21, § 7.4] and identify their universal covers [Sch21, § 7.6] [FS21, § II.2.1]. Show how to compute the global sections of $\mathcal{O}(1)$ from this [Sch21, § 7.7] [FS21, Prop. II.2.2]. Deduce that $\mathcal{I}_{S^\sharp}(1) \xrightarrow{\sim} \mathcal{O}_{X_{S,E}}$ [Sch21, Prop. 7.19] [FS21, Prop. II.2.3]. Now conclude that $\mathrm{Pic}(X_{C,E}) \simeq \mathbf{Z}$, following [Sch21, § 4.4]: First, state without proof the “ampleness of $\mathcal{O}(1)$ ” [Sch21, Thm. 4.8] [FS21, Thm. II.2.6]. Using this, define the schematic (relative) Fargues–Fontaine curves and mention the GAGA theorem [Sch21, Cor. 4.10] [FS21, Prop. II.2.7]. Lastly, explain that for $S = C$, the complement of a classical point in the schematic Fargues–Fontaine curve is the spectrum of a PID and conclude [FS21, Prop. II.2.9, Prop. II.2.10].

Talk 7 Cohomology of vector bundles on the Fargues–Fontaine curve and classification

Introduce the v -topology, following [Sch21, § 5.5] [SW20, § 17.1]. Mention that it can be viewed as an analog of the fpqc topology because any map of Huber pairs from a totally disconnected perfectoid is flat modulo a pseudouniformizer ϖ [SW20, Prop. 17.1.4]. Then discuss vector bundles on affinoid adic spaces [Sch21, Thm. 8.1] and give the sketch the proof of the v -descent properties from [Sch21, Thm. 5.14]. Next, show the vanishing and v -descent results for the cohomology of vector bundles on the Fargues–Fontaine curve from [Sch21, Prop. 8.2, Prop. 8.4] [FS21, Prop. II.2.1]. Define the Banach–Colmez spaces associated with vector bundles [Sch21, Def. 8.5] [FS21, Def. II.0.6] (see also [O16, Le Bras’ lecture] and Le Bras’ thesis for more on Banach–Colmez spaces) and describe

them [Sch21, Prop. 8.6] [FS21, § II.2.2]. Finish with a sketch of the proof of the classification of vector bundles [Sch21, § 8.3] [FS21, § II.2.4].

Talk 8 Geometric interpretation of local Tate duality and $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$

Show that $X_{C,E}$ is geometrically simply connected [Wei17, § 3.4] [FF18, § 8.6] [Lur18, Lecture 27, Thm. 1]. As a consequence, explain how the Galois cohomology of E is identified with étale cohomology of $X_{C,E}$ [Far20a, Thm. 3.7] [Lur18, Lecture 27, Cor. 3]. Show that $\text{Br}(X_{C,E}) \simeq 0$ [Far20a, Thm. 2.2] [Lur18, Lecture 27, Thm. 4] and use this to define a trace map for Galois cohomology of E [Far20a, § 3.1]. Then give a geometric reinterpretation of local Tate duality and Tate’s formula for the Euler–Poincaré characteristic in Galois cohomology [Far20a, Cor. 3.8]. In a different direction, following [Wei17, Theorem B], show that $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is the étale fundamental group of an object defined over an algebraically closed field.

Talk 9 Geometric proof of local class field theory

Explain Deligne’s geometric approach to unramified local class field theory in the function field setting [O16, Bhatt’s lecture, § 1] [G19, § 6.1]. Then explain Fargues’ geometric reformulation and proof of (ramified) local class field theory via the Fargues–Fontaine curve [Far20b] [G19, Ch. 6].

Talk 10 Weakly admissible implies admissible

Explain the proof via the Fargues–Fontaine curve that weakly admissible implies admissible [Ans20, Lecture 14] [FF18, § 10.5] [Mor19, Section 3.3] by constructing fully faithful functors from \mathbf{Q}_p -representations of G_K and filtered φ -modules to G_K -equivariant vector bundles on X . It would be nice to phrase everything in the language of adic Fargues–Fontaine curves from the seminar.

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