

# ISOMORPHISMS UP TO BOUNDED TORSION BETWEEN RELATIVE $K_0$ -GROUPS AND CHOW GROUPS WITH MODULUS

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ABSTRACT. The purpose of this note is to establish isomorphisms up to bounded torsion between relative  $K_0$ -groups and Chow groups with modulus as defined by Binda and Saito.

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## 0. INTRODUCTION

Let  $X$  be a separated regular noetherian scheme of finite Krull dimension  $d$ . As a celebrated consequence of Soulé's Riemann-Roch type formula in [So85], one has an isomorphism up to  $(d-1)!$ -power torsion

$$K_0(X) \left[ \frac{1}{(d-1)!} \right] \simeq \mathrm{CH}^*(X) \left[ \frac{1}{(d-1)!} \right]$$

between the Grothendieck group of  $X$  and the Chow group of  $X$ . The purpose of this note is to generalize the isomorphism to the relative situation, where  $X$  is equipped with an effective Cartier divisor  $D$ .

We let  $K_0(X, D)$  denote the zero-th homotopy group of the homotopy fiber of the canonical morphism  $K(X) \rightarrow K(D)$  between the  $K$ -theory spectra, and  $\mathrm{CH}^*(X|D)$  the Chow group with modulus defined by Binda and Saito in [BS17]. Here is our main theorem.

**Theorem 0.1.** *Let  $X$  be a separated regular noetherian scheme of finite Krull dimension  $d$  and  $D$  an effective Cartier divisor on  $X$ . Assume that  $D$  has an affine open neighborhood in  $X$ . Then there exists a finite descending filtration  $F^*$  on  $K_0(X, D)$  and a surjective group morphism for each integer  $p$*

$$\mathrm{cyc}: \mathrm{CH}^p(X|D) \twoheadrightarrow F^p K_0(X, D) / F^{p+1} K_0(X, D)$$

*such that its kernel is  $(p-1)!^N$ -torsion for some positive integer  $N$  depending only on  $p$ . Furthermore, the filtration  $F^*$  coincides with the  $\gamma$ -filtration on  $K_0(X, D)$  up to  $(d-1)!^M$ -torsion for some positive integer  $M$  depending only on  $d$ . In particular, there is an isomorphism*

$$K_0(X, D) \left[ \frac{1}{(d-1)!} \right] \simeq \mathrm{CH}^*(X|D) \left[ \frac{1}{(d-1)!} \right].$$

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The filtration  $F^*$  and the morphism  $\text{cyc}$  have been constructed in [Iw19] in a slightly weaker generality. If  $A$  is a regular ring, the filtration  $F^*$  on  $K_0(\Delta_A^n, \partial\Delta_A^n) \simeq K_n(A)$  coincides with the one studied by Bloch in [Bl94]<sup>1</sup>, cf. Theorem 5.7. On zero-cycles, i.e.,  $p = d$ , the morphism  $\text{cyc}$  has been studied by Binda and Krishna in [BK18]. The assumption that  $D$  has an affine open neighborhood is essential, see Example 4.8. (But maybe not in high codimension, see e.g. [BK18, Theorem 1.7] for  $p = d = 2$ .) We hope that Theorem 0.1 would be a first step for more ambitious comparison between higher relative  $K$ -groups and higher Chow groups with modulus as considered in [IKa19].

In the situation of Theorem 0.1, one might be interested in the behavior under the infinitesimal thickening of  $D$  in  $X$ . For example, our estimate on torsion makes it possible to deduce an isomorphism

$$\left(\varprojlim_m K_0(X, mD)\right) \left[\frac{1}{(d-1)!}\right] \simeq \left(\varprojlim_m \text{CH}^*(X|mD)\right) \left[\frac{1}{(d-1)!}\right],$$

where  $mD$  is the  $m$ -th infinitesimal thickening of  $D$  in  $X$ .

We conclude this introduction with an outline of the proof of Theorem 0.1. Let  $\mathcal{S}(X|D)$  be the set of all closed subsets in  $X$  not meeting  $D$  and  $\mathcal{S}(X|D, 1)$  the set of all closed subsets in  $X \times \mathbb{A}^1$  satisfying the modulus condition along  $D$ , cf. Definition 1.10. It follows easily from the definition of  $\text{CH}^*(X|D)$  that there is an exact sequence

$$\text{colim}_{Y \in \mathcal{S}(X|D, 1)} \mathcal{Z}_Y^*(X \times \mathbb{A}^1) \longrightarrow \text{colim}_{Y \in \mathcal{S}(X|D)} \text{CH}_Y^*(X) \longrightarrow \text{CH}^*(X|D) \longrightarrow 0,$$

where  $\mathcal{Z}_Y^*(-)$  is the group of cycles with supports in  $Y$  and  $\text{CH}_Y^*(-)$  is the Chow group with supports in  $Y$ . The real content of this note is to establish an analogous exact sequence for  $K$ -groups. In the second section, as Theorem 2.2, we establish the existence of an exact sequence

$$\text{colim}_{Y \in \mathcal{S}(X|D, 1)} K_0^Y(X \times \mathbb{A}^1) \longrightarrow \text{colim}_{Y \in \mathcal{S}(X|D)} K_0^Y(X) \longrightarrow K_0(X, D) \longrightarrow 0.$$

Then, from the classical rational isomorphisms between  $K_0$ -groups and Chow groups, we get a rational isomorphism between  $K_0(X, D)$  and  $\text{CH}^*(X|D)$ . The estimate on torsion is obtained by using Adams decomposition.

**Convention.** We assume that all rings are noetherian of finite Krull dimension and that all schemes are separated noetherian of finite Krull dimension. For a point  $v$  of a scheme  $X$ , we denote by  $\kappa(v)$  the residue field of  $v$ .

## 1. A PRESENTATION OF CHOW GROUP WITH MODULUS

**Chow groups with supports.** We follow [GS87, 8.1].

**Notation 1.1.** Let  $X$  be a scheme and  $p$  an integer.

- (1) We write  $X^{(p)}$  for the set of all points of codimension  $p$  in  $X$ , i.e., points  $v \in X$  whose closures in  $X$  have codimension  $p$ . We understand  $X^{(p)} = \emptyset$  if  $p < 0$ .
- (2) For a closed subset  $Y$  of  $X$ , we define  $\mathcal{Z}_Y^p(X)$  to be the free abelian group with the generators  $[V]$ , one for each  $v \in X^{(p)} \cap Y$ , with  $V$  being the closure of  $v$  in  $X$ . We write  $\mathcal{Z}^p(X) = \mathcal{Z}_X^p(X)$ .
- (3) For a closed subscheme  $D$  of pure codimension  $p$  in  $X$ , we write

$$[D] := \sum_{x_i \in D^{(0)}} \text{length}(\mathcal{O}_{D, x_i}) [D_i] \in \mathcal{Z}_D^p(X),$$

where  $D_i$  is the closure of  $x_i$  in  $X$ .

**Construction 1.2.** Let  $X$  be a scheme and  $p$  an integer. Let  $w \in X^{(p-1)}$  and write  $W$  for its closure in  $X$ . For each  $v \in X^{(p)} \cap W$ , there exists a unique group morphism  $\nu_v: \kappa(w)^\times \rightarrow \mathbb{Z}$  which sends  $a \in \mathcal{O}_{W, v} \setminus \{0\}$  to the length of  $\mathcal{O}_{W, v}/(a)$ . For  $f \in \kappa(w)^\times$ , we define

$$\text{div}(f) := \sum_{v \in X^{(p)} \cap W} \nu_v(f) [V] \in \mathcal{Z}_W^p(X).$$

<sup>1</sup>We thank Marc Levine for making us aware of this reference.

**Definition 1.3.** Let  $X$  be a scheme and  $p$  an integer. For a closed subset  $Y$  of  $X$ , we define

$$\mathrm{CH}_Y^p(X) := \mathrm{coker} \left( \bigoplus_{w \in X^{(p-1)} \cap Y} \kappa(w)^\times \xrightarrow{\mathrm{div}} \mathcal{Z}_Y^p(X) \right).$$

We write  $\mathrm{CH}^p(X) = \mathrm{CH}_X^p(X)$ .

**Definition 1.4.** Let  $X$  be a topological space with irreducible components  $\{X_i\}_{i \in I}$ . We say that  $X$  is *well-codimensional* if  $\mathrm{codim}_{X_i}(V) = \mathrm{codim}_{X_j}(V)$  for any  $i, j \in I$  and any irreducible closed subset  $V$  in  $X_i \cap X_j$ . A scheme is well-codimensional if the underlying topological space is well-codimensional.

**Lemma 1.5.** *Let  $X$  be a well-codimensional catenary scheme,  $D$  a closed subscheme of pure codimension  $r$  in  $X$  and  $p$  an integer. Then  $D$  is well-codimensional and  $D^{(p-r)} \subset X^{(p)}$ . Furthermore, the inclusion  $\iota: D \hookrightarrow X$  induces a group morphism*

$$\iota_*: \mathrm{CH}_{D \cap Y}^{p-r}(D) \rightarrow \mathrm{CH}_Y^p(X)$$

for any closed subset  $Y$  of  $X$ .

*Proof.* Let  $\{X_i\}_{i \in I}$  (resp.  $\{D_j\}_{j \in J}$ ) be the set of irreducible components of  $X$  (resp.  $D$ ). Take  $j \in J$  and  $v \in D^{(p-r)} \cap D_j$ . Then

$$\mathrm{codim}_{D_j}(v) + \mathrm{codim}_{X_i}(D_j) = \mathrm{codim}_{X_i}(v) = \mathrm{codim}_X(v)$$

for any  $i \in I$  with  $X_i \supset D_j$ . Since  $\mathrm{codim}_{X_i}(D_j) = r$  regardless of the choices of  $i, j$ , we see that  $D$  is well-codimensional and that  $\mathrm{codim}_X(v) = \mathrm{codim}_D(v) + r = p$ . Hence,  $D^{(p-r)} \subset X^{(p)}$ . The last statement is immediate from this.  $\square$

**Lemma 1.6.** *Let  $X$  be a well-codimensional catenary scheme,  $Y$  a closed subset of  $X$ ,  $D$  an effective Cartier divisor on  $X$  and  $p$  an integer. Let  $v \in X^{(p)} \cap Y \setminus D$ . Then  $[V \times_X D] \in \mathcal{Z}_{D \cap Y}^p(D)$ , where  $V$  is the closure of  $v$  in  $X$  with the reduced scheme structure.*

*Proof.* It suffices to show that  $\mathrm{codim}_D(V \times_X D) = p$ . First of all, note that  $V \times_X D$  is an effective Cartier divisor on  $V$ , and thus it is of pure codimension 1 in  $V$ . It follows that  $\mathrm{codim}_X(V \times_X D) = p + 1$ . Since  $X$  is catenary, we conclude that  $\mathrm{codim}_D(V \times_X D) = \mathrm{codim}_X(V \times_X D) - \mathrm{codim}_X(D) = p$ .  $\square$

**Construction 1.7.** Let  $X$  be a well-codimensional catenary scheme,  $Y$  a closed subset of  $X$ ,  $D$  an effective Cartier divisor on  $X$  and  $p$  an integer. We define a group morphism

$$\iota^*: \mathcal{Z}_Y^p(X) \rightarrow \mathrm{CH}_{Y \cap D}^p(D)$$

as follows, where  $\iota$  refers to the inclusion  $D \hookrightarrow X$ . For an integral closed subscheme  $V$  of codimension  $p$  in  $X$  whose support is in  $Y$ ,

$$\iota^*([V]) := \begin{cases} [V \times_X D] & \text{if } V \not\subseteq D \\ j_*[\mathcal{O}_X(D)|_V] & \text{if } V \subseteq D \end{cases} \in \mathrm{CH}_{Y \cap D}^p(D),$$

where the first equation is well-defined by Lemma 1.6 and, for the second,  $j$  refers to the inclusion  $V \hookrightarrow D$  and  $j_*: \mathrm{CH}^1(V) \rightarrow \mathrm{CH}_{Y \cap D}^p(D)$  is the push-forward ensured by Lemma 1.5.

*Remark 1.8.* It is a classical fact that the morphism  $\iota^*$  in Construction 1.7 factors through  $\mathrm{CH}_Y^p(X)$  if  $X$  is an algebraic scheme, cf. [Fu98, Chapter 2]. It would be true more generally, but we do not need such a result for our purpose.

### Chow groups with modulus.

**Notation 1.9.** We set  $\square^1 := \mathrm{Proj}(\mathbb{Z}[T_0, T_1])$  and let  $t$  be the rational coordinate  $T_0/T_1$ . We write  $\square^1 := \square^1 \setminus (t = \{\infty\})$ . For an integer  $q$  and a scheme  $X$ , we denote by  $\iota_{X,q}$  (or simply by  $\iota_q$ ) the inclusion  $X \hookrightarrow X \times \square^1$  defined by  $t = q$ .

**Definition 1.10** (Binda-Saito). Let  $X$  be a scheme and  $D$  an effective Cartier divisor on  $X$ . Let  $W$  be a closed subset of  $X \times \square^1$ . Let  $\overline{W}^N$  denote the normalization of the closure  $\overline{W}$  (with the reduced scheme-structure) of  $W$  in  $X \times \square^1$  and let  $\phi_W$  denote the composition  $\overline{W}^N \rightarrow \overline{W} \rightarrow X \times \square^1$  of the canonical morphisms. We say that  $W$  satisfies the modulus condition along  $D$  if the following inequality of Cartier divisors on  $\overline{W}^N$  holds

$$\phi_W^*(D \times \square^1) \leq \phi_W^*(X \times \{\infty\}).$$

**Lemma 1.11.** *Let  $X$  be a scheme and  $D$  an effective Cartier divisor on  $X$ . Let  $W$  be a closed subset of  $X \times \square^1$  satisfying the modulus condition along  $D$ . Then any closed subset of  $W$  satisfies the modulus condition along  $D$ .*

*Proof.* The proof for [KP12, Proposition 2.4] works, noting that every integral morphism of schemes is closed (the theorem of Cohen-Seidenberg, [EGA2, (6.1.10)] or [AM69, Theorem 5.10]).

Here we give an alternative argument which might be useful later. First, since the modulus condition is a local condition, we may assume  $X$  is affine  $X = \text{Spec}(A)$  and  $D$  is principal  $D = (f)$ . We give  $\overline{W} \subset X \times \square^1$  the reduced scheme structure and consider its restriction to the open subset  $X \times (\square^1 \setminus \{0\}) = \text{Spec}(A[1/t])$ . Let  $A[t^{-1}]/J$  be the coordinate ring of  $\overline{W} \cap \text{Spec}(A[1/t])$ . The modulus condition for  $W$  is equivalent to the condition that the element  $1/(tf)$  in the ring of total quotients of  $A[1/t]/J$  is integral over this ring, i.e., that there is a relation in  $A[1/t]/J$  of the form

$$\frac{1}{t^n} + g_1 \frac{1}{t^{n-1}} + \cdots + g_n = 0$$

with  $g_i \in f^i A[1/t]/J$ . Now let  $Y \subset W$  be any closed subset. Then we have the image of the above relation to the coordinate ring of  $\overline{Y} \cap \text{Spec}(A[1/t])$ . This implies the modulus condition for  $Y$ .  $\square$

**Notation 1.12.** Let  $X$  be a scheme,  $D$  an effective Cartier divisor on  $X$  and  $p$  an integer.

- (1)  $\mathcal{S}(X|D)$  is the set of all closed subsets of  $X$  not meeting  $D$ .
- (2)  $\mathcal{S}(X|D, 1)$  is the set of all closed subsets of  $X \times \square^1$  satisfying the modulus condition along  $D$ .
- (3)  $\mathcal{Z}^p(X|D)$  is the free abelian group with generators  $[V]$ , one for each  $v \in X^{(p)}$  whose closure  $V$  does not meet  $D$ .
- (4)  $\mathcal{Z}^p(X|D, 1)$  is the free abelian group with generators  $[W]$ , one for each  $w \in (X \times \square^1 \setminus \{0, 1\})^{(p)}$  whose closure  $W$  in  $X \times \square^1$  satisfies the modulus condition along  $D$ .

*Remark 1.13.* We remark that

$$\mathcal{Z}^p(X|D) = \text{colim}_{Y \in \mathcal{S}(X|D)} \mathcal{Z}_Y^p(X) \quad \text{and} \quad \mathcal{Z}^p(X|D, 1) \subset \text{colim}_{Y \in \mathcal{S}(X|D, 1)} \mathcal{Z}_Y^p(X \times \square^1).$$

In the latter formula, the difference consists of cycles contained in  $X \times \{0, 1\}$ .

**Definition 1.14.** Let  $X$  be a well-codimensional catenary scheme,  $D$  an effective Cartier divisor on  $X$  and  $p$  an integer. We define

$$\text{CH}^p(X|D) := \text{coker}(\mathcal{Z}^p(X|D, 1) \xrightarrow{\iota_0^* - \iota_1^*} \mathcal{Z}^p(X|D)),$$

where the morphisms  $\iota_0^*$  and  $\iota_1^*$  are well-defined by Lemma 1.6 and Remark 1.13.

**Lemma 1.15.** *Let  $X$  be a well-codimensional catenary scheme,  $D$  an effective Cartier divisor on  $X$ ,  $Y$  a closed subset of  $X$  not meeting  $D$  and  $p$  an integer. Then the canonical morphism*

$$\mathcal{Z}_Y^p(X) \rightarrow \text{CH}^p(X|D)$$

*factors through  $\text{CH}_Y^p(X)$ .*

*Proof.* Suppose we are given  $w \in X^{(p-1)} \cap Y$  and  $f \in \kappa(w)^\times$ . We have to show that  $\text{div}(f) = 0$  in  $\text{CH}^p(X|D)$ . Consider the Cartier divisor  $E$  on  $W \times \square^1$  defined by  $f + (1-f)t$ . Then  $E$  gives an element  $[E]$  in  $\mathcal{Z}^p(X|D, 1)$  because  $Y \cap D = \emptyset$  and  $\iota_0^*[E] - \iota_1^*[E] = \text{div}(f)$  in  $\mathcal{Z}^p(X|D)$ . This proves the lemma.  $\square$

**Proposition 1.16.** *Let  $X$  be a well-codimensional catenary scheme,  $D$  an effective Cartier divisor on  $X$  and  $p$  an integer. Then the sequence*

$$\operatorname{colim}_{Y \in \mathcal{S}(X|D,1)} \mathcal{Z}_Y^p(X \times \square^1) \xrightarrow{\iota_0^* - \iota_1^*} \operatorname{colim}_{Y \in \mathcal{S}(X|D)} \operatorname{CH}_Y^p(X) \xrightarrow{\epsilon} \operatorname{CH}^p(X|D) \longrightarrow 0$$

is exact. Here,  $\iota_0^*, \iota_1^*$  are the morphisms defined in Construction 1.7, and  $\epsilon$  is the canonical morphism as in Lemma 1.15.

*Proof.* We only have to show that the composite  $\epsilon \circ (\iota_0^* - \iota_1^*)$  is zero. Let  $Y \in \mathcal{S}(X|D, 1)$  and  $v \in (X \times \square^1)^{(p)} \cap Y$ . Then the closure  $V$  of  $v$  satisfies the modulus condition along  $D$  by Lemma 1.11. If  $v \notin X \times \{0, 1\}$ , then it is immediate from the definition of  $\operatorname{CH}^p(X|D)$  that  $(\epsilon \circ (\iota_0^* - \iota_1^*))([V]) = 0$ . If  $v \in X \times \{0, 1\}$ , say  $v \in X \times \{0\}$ , then  $\iota_1^*[V] = 0$  and  $\iota_0^*[V] = j_*[\mathcal{O}_{X \times \square^1}(X)|_V] = 0$  since  $X$  is a principal divisor in  $X \times \square^1$ . This completes the proof.  $\square$

*Remark 1.17.* Besides its use in the proof of Theorem 0.1, Proposition 1.16 is useful to get additional structures on Chow groups with modulus. To indicate an example, let us assume that  $X$  is smooth over a field. Then  $\operatorname{CH}_Y^*(X)$  and  $\operatorname{CH}_Y^*(X \times \square^1)$  have natural  $\operatorname{CH}^*(X)$ -module structures. Hence, by Proposition 1.16 (the left term factors through  $\operatorname{colim}_Y \operatorname{CH}_Y^p(X \times \square^1)$  under the present assumption),  $\operatorname{CH}^*(X|D)$  inherits a  $\operatorname{CH}^*(X)$ -module structure. More concretely, there is a pairing

$$\operatorname{CH}^p(X) \otimes_{\mathbb{Z}} \operatorname{CH}^q(X|D) \rightarrow \operatorname{CH}^{p+q}(X|D)$$

such that if  $V \in \mathcal{Z}^p(X)$  and  $W \in \mathcal{Z}^q(X|D)$  intersect properly then  $[Y].[Z] \in \operatorname{CH}^{p+q}(X|D)$  is given by Serre's Tor-formula.

## 2. A PRESENTATION OF RELATIVE $K_0$ -GROUP

For a scheme  $X$ , we denote by  $K(X)$  Thomason-Trobaugh's  $K$ -theory spectrum [TT90, 3.1]. The spectrum  $K(X)$  is contravariant functorial in  $X$ .

**Definition 2.1.** Let  $X$  be a scheme.

- (1) Let  $D$  be a closed subscheme of  $X$ . We define  $K(X, D)$  to be the homotopy fiber of the canonical morphism  $K(X) \rightarrow K(D)$ . For an integer  $n$ , we write  $K_n(X, D) = \pi_n K(X, D)$ .
- (2) Let  $Y$  be a closed subset of  $X$ . We define  $K^Y(X)$  to be the homotopy fiber of the canonical morphism  $K(X) \rightarrow K(X \setminus Y)$ . For an integer  $n$ , we write  $K_n^Y(X) = \pi_n K^Y(X)$ .

The goal of this section is to prove the following theorem.

**Theorem 2.2.** *Let  $X$  be a regular scheme and  $D$  an effective Cartier divisor on  $X$ . Assume that  $D$  admits an affine open neighborhood in  $X$ . Then the sequence*

$$\operatorname{colim}_{Y \in \mathcal{S}(X|D,1)} K_0^Y(X \times \square^1) \xrightarrow{\iota_0^* - \iota_1^*} \operatorname{colim}_{Y \in \mathcal{S}(X|D)} K_0^Y(X) \xrightarrow{\epsilon} K_0(X, D) \longrightarrow 0$$

is exact. Here,  $\epsilon$  denotes the obvious morphism.

The surjectivity of  $\epsilon$  has been observed in [Iw19], which we now recall.

**Lemma 2.3.** *Let  $X$  be a scheme and  $D$  a closed subscheme of  $X$ . Assume that  $D$  has an affine open neighborhood in  $X$ . Then the canonical morphism*

$$\epsilon: \operatorname{colim}_{Y \in \mathcal{S}(X|D)} K_0^Y(X) \rightarrow K_0(X, D)$$

is surjective.

*Proof.* Let  $U$  be an affine open neighborhood of  $D$  in  $X$ . Then, for each closed subset  $Z$  of  $U$  not meeting  $D$ , we have a commutative diagram

$$\begin{array}{ccccccc} K_0^{X \setminus U}(X) & \longrightarrow & K_0^{Z \cup (X \setminus U)}(X) & \longrightarrow & K_0^Z(U) & \longrightarrow & K_{-1}^{X \setminus U}(X) \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ K_0^{X \setminus U}(X) & \longrightarrow & K_0(X, D) & \longrightarrow & K_0(U, D) & \longrightarrow & K_{-1}^{X \setminus U}(X), \end{array}$$

where the rows are exact by the localization theorem [TT90, 7.4]. Hence, we may replace  $X$  by  $U$  and reduce to the case  $X$  is affine. Then the result follows from [Iw19, Lemma 3.4].  $\square$

### The rigidity.

**Lemma 2.4.** *Let  $X$  be a regular scheme and  $D$  an effective Cartier divisor. Let  $Y \in \mathcal{S}(X|D, 1)$  and denote its closure in  $X \times \overline{\square}^1$  by  $\overline{Y}$ . Assume that  $D$  admits an affine open neighborhood in  $X$ . Then the two morphisms*

$$\iota_0^*, \iota_1^*: K_0^{\overline{Y}}(X \times \overline{\square}^1) \rightarrow K_0(X, D)$$

*coincide.*

*Proof.* First of all, let us fix notation for morphisms of schemes:

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{\iota_0} \\ \xrightarrow{\iota_1} \end{array} & X \times \overline{\square}^1 \xrightarrow{q} X \xleftarrow{i} D \\ & & \downarrow p \\ & & \overline{\square}^1 \end{array}$$

where  $p, q$  are the canonical projections and  $i$  is the canonical inclusion.

According to [Iw19, Theorem 3.1],  $K_0(X, D)$  is generated by triples  $(P, \alpha, Q)$  where  $P, Q$  are perfect complexes of  $X$  and  $\alpha$  is a quasi-isomorphism  $Li^*P \xrightarrow{\sim} Li^*Q$  in the derived category of  $D$ . The morphism

$$\iota_a^*: K_0^{\overline{Y}}(X \times \overline{\square}^1) \rightarrow K_0(X, D) \quad a \in \{0, 1\}$$

sends  $[P]$  to  $[(L\iota_a^*P, 0, 0)]$ , where  $P$  is a perfect complex of  $X \times \overline{\square}^1$  whose support lies in  $\overline{Y}$ . Since  $X$  is regular,  $K_0^{\overline{Y}}(X \times \overline{\square}^1)$  is generated by coherent  $\mathcal{O}_{\overline{Y}}$ -modules. Hence, it suffices to show that, for any coherent  $\mathcal{O}_{\overline{Y}}$ -module  $\mathcal{F}$ ,

$$[(L\iota_0^*\mathcal{F}, 0, 0)] = [(L\iota_1^*\mathcal{F}, 0, 0)]$$

in  $K_0(X, D)$ . In the sequel, we denote by  $\mathcal{F}$  a coherent  $\mathcal{O}_{\overline{Y}}$ -module.

*First calculation in  $K_0(X, D)$ .* We write  $j_0$  for the inclusion  $\mathcal{O}(-1) \rightarrow \mathcal{O}_{\overline{\square}^1}$  as the ideal defining  $0 \in \overline{\square}^1$  and  $j_1$  for the inclusion as the ideal defining  $1 \in \overline{\square}^1$ . Then we have an exact triangle

$$Lp^*\mathcal{O}(-1) \otimes_{X \times \overline{\square}^1}^L \mathcal{F} \xrightarrow{p^*j_a} \mathcal{F} \longrightarrow \iota_{a*}L\iota_a^*\mathcal{F} \xrightarrow{+}$$

of perfect complexes of  $X \times \overline{\square}^1$  for  $a \in \{0, 1\}$ . By applying  $Rq_*$  to the exact triangle, we get an equality

$$[(L\iota_a^*\mathcal{F}, 0, 0)] = -Rq_*[(Lp^*\mathcal{O}(-1) \otimes_{X \times \overline{\square}^1}^L \mathcal{F}, p^*j_a, \mathcal{F})]$$

in  $K_0(X, D)$ . Here, the right hand side means a triple to which  $Rq_*$  is applied term-wise.

Recall that we have fixed a rational coordinate  $t$  of  $\overline{\square}^1$ . We set  $\theta := (t - 1)/t$ . Then we have a commutative diagram (the vertical arrow is defined after restricting to  $\overline{\square}^1 \setminus \{0\}$ )

$$\begin{array}{ccc} & & \mathcal{O}_{\overline{\square}^1} \\ & \nearrow^{j_0} & \vdots \\ \mathcal{O}(-1) & & \text{multiplication by } \theta \\ & \searrow_{j_1} & \vdots \\ & & \mathcal{O}_{\overline{\square}^1}. \end{array}$$

Since  $Y$  satisfies the modulus condition, the support of  $\overline{Y} \cap (D \times \overline{\square}^1)$  is contained in  $D \times \{\infty\}$ , and thus the multiplication by  $\theta$  on  $\mathcal{O}_{\overline{Y}}$ -modules makes sense in a neighborhood of  $\overline{Y} \cap (D \times \overline{\square}^1)$ . It follows from the above diagram that

$$[(Lp^* \mathcal{O}(-1) \otimes_{X \times \overline{\square}^1}^L \mathcal{F}, p^* j_0, \mathcal{F})] + [(\mathcal{F}, \theta, \mathcal{F})] = [(Lp^* \mathcal{O}(-1) \otimes_{X \times \overline{\square}^1}^L \mathcal{F}, p^* j_1, \mathcal{F})].$$

Hence, it remains to show that

$$Rq_*[(\mathcal{F}, \theta, \mathcal{F})] = 0$$

in  $K_0(X, D)$ .

*Adic filtration on  $\mathcal{F}$ .* Let  $\text{Fil}^* \mathcal{F}$  be the adic filtration on  $\mathcal{F}$  with respect to the ideal defining  $D \times \{\infty\}$  in  $X \times \overline{\square}^1$ . Since  $(1/t) \text{Fil}^l \mathcal{F} \subset \text{Fil}^{l+1} \mathcal{F}$  in a neighborhood of  $\overline{Y} \cap (D \times \overline{\square}^1)$ , the  $\theta$  acts on  $\text{Fil}^l \mathcal{F} / \text{Fil}^{l+1} \mathcal{F}$  as the identity in such a neighborhood. Since  $[(P, \text{id}, P)] \in K_0(X, D)$  is zero in general, we see that

$$[(\text{Fil}^l \mathcal{F} / \text{Fil}^{l+1} \mathcal{F}, \theta, \text{Fil}^l \mathcal{F} / \text{Fil}^{l+1} \mathcal{F})] = 0$$

for all  $l \geq 0$ . Hence, we are reduced to showing that  $Rq_*[(\text{Fil}^l \mathcal{F}, \theta, \text{Fil}^l \mathcal{F})] = 0$  for some  $l \geq 0$ . We show that  $\theta$  acts on  $Li^* Rq_* \text{Fil}^l \mathcal{F}$  as the identity for sufficiently large  $l$ .

Take an affine open neighborhood  $U$  of  $D$  in  $X$  such that  $\overline{Y}_U := \overline{Y} \times_X U$  misses  $X \times \{0, 1\}$  and that the restriction  $q|_{\overline{Y}_U} : \overline{Y}_U \rightarrow U$  is finite. We set notation:

$$A = \Gamma(U, \mathcal{O}) \quad A/I = \Gamma(D, \mathcal{O}) \quad A[1/t]/J = \Gamma(\overline{Y}_U, \mathcal{O}) \quad M = \mathcal{F}(\overline{Y}_U)$$

The filtration  $\text{Fil}^* \mathcal{F}$  on  $\mathcal{F}$  descends to a filtration  $\text{Fil}^* M$  on  $M$ , which is identified with the  $(I, 1/t)$ -adic filtration. Observe that  $Li^* Rq_* \text{Fil}^l \mathcal{F} = (Li^* \text{Fil}^l M) \sim$ .

We claim that there exists  $n \geq 0$  such that (1)  $\text{Fil}^{l+1} M = I \text{Fil}^l M$  and (2)  $H_k(Li^* \text{Fil}^l M) = 0$  for all  $l \geq n$  and  $k > 0$ . If we admit the claim, then

$$Li^* \text{Fil}^l M \stackrel{(2)}{=} \text{Fil}^l M / I \text{Fil}^l M \stackrel{(1)}{=} \text{Fil}^l M / \text{Fil}^{l+1} M$$

on which we know that  $\theta$  acts as the identity. The claim is a local question on  $\text{Spec } A$ , and thus we may assume that  $I$  is principal,  $I = (f)$  with  $0 \neq f \in A$ . By the modulus condition, we have a relation in  $A[1/t]/J$  of the form

$$\frac{1}{t^n} + g_1 \frac{1}{t^{n-1}} + \cdots + g_{n-1} \frac{1}{t} + g_n = 0$$

for some  $n \geq 0$  and  $g_k \in f^k A[1/t]/J$  with  $1 \leq k \leq n$ . Repeated application of this relation gives

$$\text{Fil}^l M = f^l M + \frac{f^{l-1}}{t} M + \cdots + \frac{f^{l-(n-1)}}{t^{n-1}} M$$

for  $l \geq n$ . In particular,  $\text{Fil}^{l+1} M = f \text{Fil}^l M$ . Furthermore, since the  $f$ -power torsion of the finitely generated module  $M$  has a bounded exponent,  $\text{Fil}^l M = f^{l-n} \text{Fil}^n M$  has no  $f$ -power torsion for  $l \gg n$ . This proves the claim.  $\square$

**Corollary 2.5.** *Under the situation in Theorem 2.2,  $\epsilon \circ (\iota_0^* - \iota_1^*) = 0$ .*

*Proof.* Since  $X$  is regular, the restriction morphism

$$K_0^{\overline{Y}}(X \times \square^1) \rightarrow K_0^Y(X \times \square^1)$$

is surjective, where  $Y \in \mathcal{S}(X|D, 1)$  and  $\overline{Y}$  is the closure of  $Y$  in  $X \times \square^1$ . Hence, the result follows from Lemma 2.4.  $\square$

*Remark 2.6.* When the modulus  $D$  is  $K_1$ -regular, Lemma 2.4 and Corollary 2.5 immediately follow from the isomorphism  $K_0(X, D) \simeq K_0(X \times \square^1, D \times \square^1)$ . In fact, the modulus condition does not play any essential role here, see §5 for more about this case.

**End of the proof.**

**Lemma 2.7.** *Let  $X$  be a scheme and  $D$  an effective Cartier divisor on  $X$  admitting an affine open neighborhood in  $X$ . Suppose we are given  $Z \in \mathcal{S}(X|D)$  and  $\alpha \in K_1(X \setminus Z)$  whose restriction to  $K_1(D)$  is zero. Then there exist  $W \in \mathcal{S}(X|D, 1)$  and  $\beta \in K_1(X \times \square^1 \setminus W)$  such that*

$$\beta|_{t=0} - \beta|_{t=1} = \alpha \quad \text{in} \quad \operatorname{colim}_{Y \in \mathcal{S}(X|D)} K_1(X \setminus Y).$$

*Proof.* By enlarging  $Z$  if necessary, we may assume that  $X \setminus Z$  is affine. Set  $U = X \setminus Z$ . Take a representative of  $\alpha$  in  $\operatorname{GL}_n(U)$ , which we also denote by  $\alpha$ . By our assumption, the restriction of  $\alpha$  to matrices over  $D$  is in the group  $E_m(D)$  of elementary matrices for some  $m \geq n$ . Take a lift  $\alpha' \in E_m(U)$  of  $\alpha|_D$ . Then  $\alpha = \alpha' + \epsilon$  in  $\operatorname{GL}_m(U)$ , where  $\epsilon$  is a matrix whose entries are all in the ideal  $I$  defining  $D$ . We define an  $(m \times m)$ -matrix over  $U \times \square^1$  by

$$\alpha(t) := \alpha' + (1 - t)\epsilon.$$

Then the determinant  $\det(\alpha(t))$  is an admissible polynomial for  $D$  in the sense [BS17, §4], and thus its zero locus  $W = V(\det(\alpha(t)))$  satisfies the modulus condition along  $D$ . Lastly, by definition,  $\alpha(t)$  gives an element  $\beta$  in  $K_1(X \times \square^1 \setminus W)$  and it satisfies the desired formula.  $\square$

*Proof of Theorem 2.2.* By Lemma 2.3 and Corollary 2.5, it remains to show the exactness at the middle term. Suppose we are given  $Z \in \mathcal{S}(X|D)$  and  $\alpha \in K_0^Z(X)$  which is killed in  $K_0(X, D)$  along the obvious morphism. Look at the commutative diagram with exact rows

$$\begin{array}{ccccc} K_1(X \setminus Z, D) & \longrightarrow & K_1(X \setminus Z) & \longrightarrow & K_1(D) \\ & & \downarrow \text{boundary} & & \downarrow \text{boundary} \\ K_1(X \setminus Z, D) & \longrightarrow & K_0^Z(X) & \longrightarrow & K_0(X, D). \end{array}$$

It follows that there exists a lift  $\alpha' \in K_1(X \setminus Z)$  of  $\alpha$  along the boundary morphism whose restriction to  $K_1(D)$  is zero. Hence, by Lemma 2.7, we find an element  $\beta'$  in the group at the left upper corner of the commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_{Y \in \mathcal{S}(X|D, 1)} K_1(X \times \square^1 \setminus Y) & \xrightarrow{\iota_0^* - \iota_1^*} & \operatorname{colim}_{Y \in \mathcal{S}(X|D)} K_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \operatorname{colim}_{Y \in \mathcal{S}(X|D, 1)} K_0^Y(X \times \square^1) & \xrightarrow{\iota_0^* - \iota_1^*} & \operatorname{colim}_{Y \in \mathcal{S}(X|D)} K_0^Y(X) \xrightarrow{\epsilon} K_0(X, D) \end{array}$$

such that  $\iota_0^* \beta' - \iota_1^* \beta' = \alpha'$  in the upper middle group. This proves the exactness of the lower sequence.  $\square$

## 3. ADAMS DECOMPOSITION

**Notation 3.1.** For  $i \geq 0$ , we set

$$w_i = \begin{cases} 1 & \text{if } i \text{ is zero} \\ 2 & \text{if } i \text{ is odd} \\ \text{denominator of } |B_{i/2}|/2i & \text{if } i \text{ is positive even} \end{cases}$$

where  $B_n$  denotes the  $n$ -th Bernoulli number.

**Lemma 3.2.**

- (i) If a prime  $p$  divides  $w_i$ , then  $(p-1)$  divides  $i$ . The converse is true if  $i$  is even.
- (ii) For  $i \geq 1$  and for  $N$  large enough than  $i$ ,

$$w_i = \gcd_{k \geq 2} k^N (k^i - 1).$$

*Proof.* See [MS74, Appendix B]. □

**Definition 3.3.** Let  $N$  be a positive integer. We say that a morphism  $f: A \rightarrow B$  of abelian groups is an  $N$ -monomorphism (resp.  $N$ -epimorphism) if the kernel (resp. cokernel) of  $f$  is killed by  $N$ . We say that  $f$  is an  $N$ -isomorphism if it is an  $N$ -monomorphism and an  $N$ -epimorphism.

**Proposition 3.4.** Let  $A$  be an abelian group equipped with endomorphisms  $\psi^k$  for  $k > 0$  which commute with each other. Suppose that there is a finite filtration

$$A = F^i \supset F^{i+1} \supset \dots \supset F^j \supset F^{j+1} = 0$$

with  $0 \leq i \leq j$  consisting of subgroups preserved by  $\psi^k$  such that  $\psi^k$  acts on  $F^p/F^{p+1}$  by the multiplication by  $k^p$ . For  $p \geq 0$ , let  $A^{(p)}$  be the subgroup of  $A$  consisting of elements  $x \in A$  such that  $\psi^k x = k^p x$  for all  $k > 0$ . Then:

- (i) For  $i \leq p \leq j$ ,  $(\prod_{q=i}^{p-1} w_{p-q})A^{(p)}$  is in  $F^p$  and the induced morphism

$$\prod_{q=i}^{p-1} w_{p-q}: A^{(p)} \rightarrow F^p/F^{p+1}$$

is a  $(\prod_{q=i}^j w_{|p-q|})$ -isomorphism.

- (ii) The canonical morphism

$$\bigoplus_{p=i}^j A^{(p)} \rightarrow A$$

is a  $(\prod_{i \leq q, p \leq j} w_{|p-q|})$ -isomorphism.

*Proof.* We follow [So85, 2.8]. Take integers  $A_{pqk}$  ( $p \neq q$ ) so that

$$w_{|p-q|} = \sum_{k>1} A_{pqk} (k^p - k^q).$$

We fix  $i \leq p \leq j$ . The morphism

$$\Phi_p := \prod_{q=i}^{p-1} \left( \sum_{k>1} A_{pqk} (\psi^k - k^q) \right)$$

sends  $A$  to  $F^p$ , and on  $A^{(p)}$  it is the multiplication by  $\prod_{q=i}^{p-1} w_{p-q}$ . On the other hand, the morphism

$$\Psi_p := \prod_{q=p+1}^j \left( \sum_{k>1} A_{pqk} (\psi^k - k^q) \right)$$

sends  $F^p$  to  $A^{(p)}$ , and modulo  $F^{p+1}$  it is the multiplication by  $\prod_{q=p+1}^j w_{q-p}$ . Moreover, this morphism kills  $F^{p+1}$ . To summarize,  $(\prod_{q=i}^{p-1} w_{p-q})A^{(p)}$  is in  $F^p$  and the induced morphism

$$\prod_{q=i}^{p-1} w_{p-q}: A^{(p)} \rightarrow F^p/F^{p+1}$$

is a  $(\prod_{q=i}^j w_{|p-q|})$ -isomorphism.

Next, we prove (ii). Let us consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{p=i+1}^q A^{(p)} \cap F^{i+1} & \longrightarrow & \bigoplus_{p=i}^q A^{(p)} & \longrightarrow & \left( \bigoplus_{p=i+1}^q A^{(p)}/A^{(p)} \cap F^{i+1} \right) \oplus A^{(i)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F^{i+1} & \longrightarrow & A & \longrightarrow & A/F^{i+1} \longrightarrow 0 \end{array}$$

with exact rows. The left vertical arrow is a  $(\prod_{i+1 \leq p, q \leq j} w_{|p-q|})$ -isomorphism by induction. Since  $\Phi_{i+1}$  sends  $A$  to  $F^{i+1}$  and it is the multiplication by  $w_{p-i}$  on  $A^{(p)}$ ,  $\bigoplus_{p=i+1}^q A^{(p)}/A^{(p)} \cap F^{i+1}$  is killed by  $\prod_{p=i+1}^j w_{p-i}$ . Combining it with (i), we see that the right vertical arrow is a  $(\prod_{p=i}^j w_{p-i})^2$ -isomorphism. Consequently, the middle vertical arrow is a  $(\prod_{i \leq p, q \leq j} w_{|p-q|})$ -isomorphism.  $\square$

**Example I:  $\gamma$ -filtration.** We refer to [AT69] for the definition of (non-unital) special  $\lambda$ -rings,  $\gamma$ -filtrations and Adams operations.

**Lemma 3.5.** *Let  $I$  be a non-unital special  $\lambda$ -ring. Assume that the  $\gamma$ -filtration on  $I$  is finite. Then the  $\gamma$ -filtration on  $I$  together with the Adams operations satisfies the condition of Proposition 3.4.*

*Proof.* This follows from [loc. cit., Proposition I.5.3].  $\square$

**Definition 3.6.** Let  $X$  be a scheme. We define

$$\tilde{K}(X) := \mathbb{H}_{\text{Zar}}(X, B\text{GL}^+)$$

the global sections over  $X$  of a Zariski-fibrant replacement of (a functorial model of)  $B\text{GL}^+$ . Let  $Y$  be a closed subset of  $X$  and  $D$  a closed subscheme of  $X$ . We define  $\tilde{K}^Y(X, D)$  to be the iterated homotopy fiber of the square

$$\begin{array}{ccc} \tilde{K}(X) & \longrightarrow & \tilde{K}(D) \\ \downarrow & & \downarrow \\ \tilde{K}(X \setminus Y) & \longrightarrow & \tilde{K}(D \setminus Y). \end{array}$$

For a non-negative integer  $n$ , we write  $\tilde{K}_n^Y(X, D) := \pi_n \tilde{K}^Y(X, D)$ . When  $D = \emptyset$  (resp.  $Y = X$ ), we write  $\tilde{K}^Y(X)$  (resp.  $\tilde{K}(X, D)$ ) for  $\tilde{K}^Y(X, D)$ .

**Remark 3.7.** Informally,  $\tilde{K}(X)$  is the first  $\gamma$ -filtration of  $\Omega^\infty K(X)$ . In fact, since Zariski locally  $\Omega^\infty K \simeq \mathbb{Z} \times B\text{GL}^+$ , we have

$$K_n(X) \simeq \begin{cases} \mathbb{Z} \oplus \tilde{K}_0(X) & \text{if } n = 0 \\ \tilde{K}_n(X) & \text{if } n \geq 1. \end{cases}$$

**Lemma 3.8.** *Let  $X$  be a scheme,  $Y$  a closed subset of  $X$  and  $D$  a closed subscheme of  $X$ . Then  $\tilde{K}_n^Y(X, D)$  is naturally a special  $\lambda$ -ring for each  $n \geq 0$ . The first grading of the  $\gamma$ -filtration is*

$$F_\gamma^1 \tilde{K}_n^Y(X, D) / F_\gamma^2 \tilde{K}_n^Y(X, D) \simeq \begin{cases} \mathbb{H}_{\text{Zar}, Y}^1((X, D), \mathcal{O}^\times) & n = 0 \\ \mathbb{H}_{\text{Zar}, Y}^0((X, D), \mathcal{O}^\times) & n = 1 \\ 0 & n > 1 \end{cases}$$

where  $\mathbb{H}_{\text{Zar}, Y}^*((X, D), \mathcal{O}^\times)$  denotes the homology of the iterated homotopy fiber of the square

$$\begin{array}{ccc} \mathbb{H}_{\text{Zar}}(X, \mathcal{O}^\times) & \longrightarrow & \mathbb{H}_{\text{Zar}}(D, \mathcal{O}^\times) \\ \downarrow & & \downarrow \\ \mathbb{H}_{\text{Zar}}(X \setminus Y, \mathcal{O}^\times) & \longrightarrow & \mathbb{H}_{\text{Zar}}(D \setminus Y, \mathcal{O}^\times). \end{array}$$

*Proof.* Refer to [Le97, Corollary 5.6] for the fact  $\tilde{K}_n^Y(X, D)$  is a special  $\lambda$ -ring. The first grading is calculated by the determinant  $\det: BGL^+ \rightarrow B\mathcal{O}^\times$ . Indeed, we have  $\gamma^1 + \gamma^2 + \dots = 0$  in  $SK_*^Y(X, D) = \mathbb{H}_{\text{Zar}, Y}^*((X, D), BSL^+)$  as in [So85, p524], and thus  $F_\gamma^2 SK_*^Y(X, D) = SK_*^Y(X, D)$ .  $\square$

### Example II: coniveau filtration.

**Definition 3.9.** Let  $X$  be a scheme and  $Y$  a closed subset of  $X$ . For each  $p \geq 0$ , we define

$$F^p K_0^Y(X) := \operatorname{colim}_Z \operatorname{image}(K_0^Z(X) \rightarrow K_0^Y(X))$$

where  $Z$  runs over all closed subsets of  $Y$  whose codimension in  $X$  are greater or equal to  $p$ . We call the filtration the *coniveau filtration*. We write  $\operatorname{Gr}^p K_0^Y(X)$  for the  $p$ -th grading of the coniveau filtration.

**Lemma 3.10.** Let  $X$  be a regular scheme of dimension  $d$  and  $Y$  a closed subset of  $X$  of codimension  $p$ . Then the coniveau filtration

$$K_0^Y(X) = F^p \supset F^{p+1} \supset \dots \supset F^d \supset F^{d+1} = 0$$

together with the Adams operations satisfies the condition of Proposition 3.4.

*Proof.* First, we prove the case  $Y$  is a closed point of  $X$ . Let  $j$  be the inclusion  $Y \hookrightarrow X$  and  $j_*$  the pushforward  $K_0(Y) \rightarrow K_0^Y(X)$ . Then, by [So85, Théorème 3], we have  $\psi^k \circ j_* = k^d(j_* \circ \psi^k)$ . Since  $\psi^k$  is the identity on  $K_0(Y) \simeq \mathbb{Z}$  and  $j_*$  is an isomorphism, we conclude that  $\psi^k$  acts by the multiplication by  $k^d$  on  $K_0^Y(X)$ . This proves the case  $Y$  is a closed point.

We prove the remaining case by descending induction on  $p$ . We have seen the case  $p = d$ . Let  $p < d$ . If  $q > p$ , then the canonical morphism

$$\operatorname{colim}_Z \operatorname{Gr}^q K_0^Z(X) \rightarrow \operatorname{Gr}^q K_0^Y(X),$$

where  $Z$  runs over all closed subsets of  $Y$  whose codimension in  $X$  are greater or equal to  $p + 1$ , is surjective. By induction, the Adams operation  $\psi^k$  acts by the multiplication by  $k^q$  on the left term, and so on the right. It remains to show that the Adams operation  $\psi^k$  acts by the multiplication by  $k^p$  on  $\operatorname{Gr}^p K_0^Y(X) = K_0^Y(X)/F^{p+1}K_0^Y(X)$ . This follows from the exact sequence [GS87, Lemma 5.2]

$$0 \longrightarrow F^{p+1}K_0^Y(X) \longrightarrow K_0^Y(X) \longrightarrow \bigoplus_{y \in Y \cap X^{(p)}} K_0^y(X_y) \longrightarrow 0.$$

Note that the Adams operations act on the sequence and we have seen that  $\psi^k$  acts on the right term by the multiplication by  $k^p$ . This completes the proof.  $\square$

### Example III: relative coniveau filtration.

**Definition 3.11.** Let  $X$  be a scheme and  $D$  a closed subscheme of  $X$ . For each  $p \geq 0$ , we define

$$F^p K_0(X, D) := \operatorname{colim}_Z \operatorname{image}(K_0^Z(X) \rightarrow K_0(X, D))$$

where  $Z$  runs over all closed subsets in  $X$  not meeting  $D$  of codimension greater or equal to  $p$ . We call the filtration the *relative coniveau filtration*. We write  $\operatorname{Gr}^p K_0(X, D)$  for the  $p$ -th grading of the relative coniveau filtration.

**Lemma 3.12.** *Let  $X$  be a scheme of dimension  $d$  with an ample family of line bundles and  $D$  a closed subscheme of  $X$ . Assume that  $X \setminus D$  is regular and that  $D$  has an affine open neighborhood in  $X$ . Then the relative coniveau filtration*

$$K_0(X, D) = F^0 \supset F^1 \supset \dots \supset F^d \supset F^{d+1} = 0$$

*together with the Adams operations satisfies the condition of Proposition 3.4.*

*Proof.* By Lemma 2.3,  $F^0 K_0(X, D) = K_0(X, D)$ . By definition, the canonical morphism

$$\operatorname{colim}_{Y \in \mathcal{S}(X|D)} \operatorname{Gr}^p K_0^Y(X) \rightarrow \operatorname{Gr}^p K_0(X, D)$$

is surjective. The morphism is compatible with the Adams operations, and thus the result follows from the fact  $K_0^Y(X) \simeq K_0^Y(X \setminus D)$  and Lemma 3.10.  $\square$

#### 4. PROOF OF THEOREM 0.1

##### Cycle class morphisms with supports.

**Definition 4.1.** Let  $X$  be a regular scheme,  $Y$  a closed subset of  $X$  and  $p$  an integer. The *cycle class morphism* is a group morphism

$$\operatorname{cyc}: \mathcal{Z}_Y^p(X) \rightarrow K_0^Y(X)$$

defined by sending the closure  $V$  of a point  $v \in X^{(p)} \cap Y$  to  $[\mathcal{O}_V]$ . Note that  $\mathcal{O}_V$  is a perfect complex since  $X$  is regular.

**Theorem 4.2** (Gillet-Soulé). *Let  $X$  be a regular scheme,  $Y$  a closed subset of  $X$  and  $p$  an integer. Then the cycle class morphism induces a surjective group morphism*

$$\operatorname{cyc}: \operatorname{CH}_Y^p(X) \twoheadrightarrow F^p K_0^Y(X) / F^{p+1} K_0^Y(X)$$

*and its kernel is  $(\prod_{i=1}^{p-2} \omega_i)$ -torsion.*

*Proof.* This is essentially a consequence of results in [So85] as observed in [GS87, Theorem 8.2]. Since our formulation claims a little bit stronger than the original one, we give a sketch of the proof here.

Consider the Gersten-Quillen spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)} \cap Y} K_{-p-q}(k(x)) \Rightarrow K_{-p-q}^Y(X).$$

The Adams operations act on this spectral sequence, and by the Riemann-Roch type formula [So85, Théorème 3]  $\psi^k$  acts by the multiplication by  $k^{p+i}$  on  $E_r^{p,-p-i}$  for  $i = 0, 1$  and  $r \geq 1$ . It follows that the differential  $d_r^p: E_r^{p-r,-p+r-1} \rightarrow E_r^{p,-p}$  is killed by  $w_{r-1}$  for  $p \geq r > 1$  (use the presentation of  $w_{r-1}$  as in Lemma 3.2 (ii)). In fact,  $d_r^r = 0$  for  $r > 1$ , because

$$E_2^{0,-1} \simeq \bigoplus_{x \in X^{(0)} \cap Y} \mathcal{O}(\overline{\{x\}})^\times \simeq E_\infty^{0,-1}.$$

Consequently, the kernel of the canonical surjection  $E_2^{p,-p} \twoheadrightarrow E_\infty^{p,-p}$  is killed by  $\prod_{i=1}^{p-2} w_i$ . On the other hand, we have  $E_2^{p,-p} \simeq \operatorname{CH}_Y^p(X)$  and  $E_\infty^{p,-p} \simeq \operatorname{Gr}^p K_0^Y(X)$ . Hence, we get the result.  $\square$

**Lemma 4.3.** *Let  $X$  be a regular scheme,  $Y$  a closed subset of  $X$ ,  $D$  a principal effective Cartier divisor on  $X$  and  $p$  an integer. We denote the inclusion  $D \hookrightarrow X$  by  $\iota$ . Then the diagram*

$$\begin{array}{ccc} K_0^Y(X) & \xrightarrow{\iota^*} & K_0^{Y \cap D}(D) / F^{p+1} K_0^{Y \cap D}(D) \\ \operatorname{cyc} \uparrow & & \uparrow \operatorname{cyc} \\ \mathcal{Z}_Y^p(X) & \xrightarrow{\iota^*} & \operatorname{CH}_{Y \cap D}^p(D) \end{array}$$

*commutes. Here, the bottom horizontal map  $\iota^*$  is the one defined in Construction 1.7.*

*Proof.* Suppose that we are given  $v \in X^{(p)} \cap Y$  and denote its closure in  $X$  by  $V$ . If  $V \not\subseteq D$ , then the commutativity is clear, i.e.,  $(\iota^* \circ \text{cyc})[V] = (\text{cyc} \circ \iota^*)[V]$ . Suppose that  $V \subseteq D$ . Then  $\iota^*[V] = 0$  in  $\text{CH}_{Y \cap D}^p(D)$ . Also,  $\iota^*(\mathcal{O}_V) = [\mathcal{O}_V \xrightarrow{f} \mathcal{O}_V] = 0$  in  $K_0^{Y \cap D}(D)$ , where  $f$  denotes the defining equation of  $D$ . This proves the lemma.  $\square$

**Corollary 4.4.** *Under the situation in Lemma 4.3, the restriction morphism  $K_0^Y(X) \rightarrow K_0^{Y \cap D}(D)$  preserves the coniveau filtration.*

**On codimension one.**

**Lemma 4.5.** *Let  $X$  be a regular scheme and  $Y$  a closed subset of  $X$  not containing any irreducible components of  $X$ . Then there are natural isomorphisms*

$$\begin{aligned} K_0^Y(X) &\simeq \text{CH}_Y^1(X) \oplus F^2 K_0^Y(X), \\ \text{CH}_Y^1(X) &\simeq \mathbb{H}_{\text{Zar}, Y}^1(X, \mathcal{O}^\times) \quad \text{and} \quad F^2 K_0^Y(X) \simeq F_\gamma^2 K_0^Y(X). \end{aligned}$$

*Proof.* Since  $K_0^Y(X) = F^1 K_0^Y(X)$  by our assumption, we get an isomorphism

$$\text{cyc}: \text{CH}_Y^1(X) \xrightarrow{\sim} K_0^Y(X)/F^2 K_0^Y(X)$$

by Theorem 4.2. Since  $\text{CH}_Y^1(X)$  is the free abelian group generated by the irreducible components of  $Y$  of codimension one in  $X$ , the above isomorphism factors through  $K_0^Y(X)$ . This proves the first isomorphism. The second isomorphism follows from the quasi-isomorphism  $\mathcal{Z}^1(X, \bullet) \simeq \mathcal{O}^\times[1]$  ([BI86, Theorem 6.1]). The last isomorphism follows from the first two isomorphisms and Lemma 3.8.  $\square$

**Corollary 4.6.** *Let  $X$  be a regular scheme and  $D$  an effective Cartier divisor on  $X$ . Assume that  $D$  has an affine open neighborhood in  $X$ . Then there are exact sequences*

$$\begin{aligned} \text{colim}_{Y \in \mathcal{S}(X|D, 1)} F^2 K_0^Y(X \times \square^1) &\xrightarrow{\iota_0^* - \iota_1^*} \text{colim}_{Y \in \mathcal{S}(X|D)} F^2 K_0^Y(X) \xrightarrow{\epsilon} F^2 K_0(X, D) \longrightarrow 0 \\ \text{colim}_{Y \in \mathcal{S}(X|D, 1)} \text{Gr}^1 K_0^Y(X \times \square^1) &\xrightarrow{\iota_0^* - \iota_1^*} \text{colim}_{Y \in \mathcal{S}(X|D)} \text{Gr}^1 K_0^Y(X) \xrightarrow{\epsilon} \text{Gr}^1 K_0(X, D) \longrightarrow 0 \end{aligned}$$

and the same for the  $\gamma$ -filtration.

*Proof.* The morphisms are well-defined by Corollary 4.4. To see the exactness, we may assume that  $X$  is connected and that  $D$  is not empty. Then, by Lemma 4.5,  $F_{(\gamma)}^2 K_0^Y(X)$  and  $F_{(\gamma)}^2 K_0^Y(X \times \square^1)$  in the sequences are direct summands of  $K_0^Y(X)$  and  $K_0^Y(X \times \square^1)$  respectively. Hence, the result follows from Theorem 2.2.  $\square$

**Theorem 4.7.** *Let  $X$  be a regular scheme and  $D$  an effective Cartier divisor on  $X$ . Assume that  $D$  has an affine open neighborhood in  $X$ . Then there are natural isomorphisms*

$$\text{CH}^1(X|D) \simeq \text{Gr}^1 K_0(X, D) \simeq \text{Gr}_\gamma^1 K_0(X, D) \simeq \text{Pic}(X, D).$$

*Proof.* The first isomorphism follows from the commutative diagram

$$\begin{array}{ccccc} \text{colim}_{Y \in \mathcal{S}(X|D, 1)} \text{Gr}^1 K_0^Y(X \times \square^1) & \xrightarrow{\iota_0^* - \iota_1^*} & \text{colim}_{Y \in \mathcal{S}(X|D)} \text{Gr}^1 K_0^Y(X) & \xrightarrow{\epsilon} & \text{Gr}^1 K_0(X, D) \longrightarrow 0 \\ \text{cyc} \uparrow & & \text{cyc} \uparrow \simeq & & \uparrow \\ \text{colim}_{Y \in \mathcal{S}(X|D, 1)} \mathcal{Z}_Y^1(X \times \square^1) & \xrightarrow{\iota_0^* - \iota_1^*} & \text{colim}_{Y \in \mathcal{S}(X|D)} \text{CH}_Y^1(X) & \xrightarrow{\epsilon} & \text{CH}^1(X|D) \longrightarrow 0. \end{array}$$

This is indeed commutative by Lemma 4.3, and the rows are exact by Proposition 1.16 and Corollary 4.6. The middle vertical arrow is an isomorphism by Theorem 4.2.

The second isomorphism follows from Lemma 4.5 and Corollary 4.6. The last isomorphism has been observed in Lemma 3.8.  $\square$

*Example 4.8.* Let  $k$  be a field,  $X = \mathbb{P}_k^1 \times_k \mathbb{P}_k^1$  and  $D = \mathbb{P}_k^1$ , which we regard as a Cartier divisor on  $X$  by the diagonal embedding. Then  $\mathrm{CH}^1(X|D) = 0$ , but  $\mathrm{Gr}_\gamma^1 K_0(X, D) = \mathrm{Pic}(X, D) = \mathbb{Z}$ .

### On higher codimension.

**Lemma 4.9.** *Let  $X$  be a regular scheme,  $D$  an effective Cartier divisor on  $X$  and  $p$  an integer. Assume that  $D$  has an affine open neighborhood in  $X$ . Suppose we are given*

$$\alpha \in \ker \left( \mathrm{colim}_{Y \in \mathcal{S}(X|D)} \mathrm{Gr}^p K_0^Y(X) \xrightarrow{\epsilon} \mathrm{Gr}^p K_0(X, D) \right).$$

*Then there exists  $\beta \in \mathrm{Gr}^p K_0^W(X \times \square^1)$  for some  $W \in \mathcal{S}(X|D, 1)$  such that*

$$\iota_0^* \beta - \iota_1^* \beta = \left( \prod_{i=2}^{p-1} w_{p-i} \right) \left( \prod_{2 \leq i, j \leq p} w_{|i-j|} \right)^2 \alpha.$$

*Proof.* We may assume that  $D$  is non-empty. The case  $p \leq 1$  is true by Theorem 2.2 and Corollary 4.6. Let  $p > 1$ . We consider the diagram

$$\begin{array}{ccc} \mathrm{colim}_{Y \in \mathcal{S}(X|D, 1)} \bigoplus_{i=2}^p (F^2 K_0^Y(X \times \square^1)/F^{p+1})^{(i)} & \longrightarrow & \mathrm{colim}_{Y \in \mathcal{S}(X|D)} \bigoplus_{i=2}^p (F^2 K_0^Y(X)/F^{p+1})^{(i)} \\ \downarrow & & \downarrow \\ \mathrm{colim}_{Y \in \mathcal{S}(X|D, 1)} F^2 K_0^Y(X \times \square^1)/F^{p+1} & \longrightarrow & \mathrm{colim}_{Y \in \mathcal{S}(X|D)} F^2 K_0^Y(X)/F^{p+1} \\ \uparrow & & \uparrow \\ \mathrm{colim}_{Y \in \mathcal{S}(X|D, 1)} \mathrm{Gr}^p K_0^Y(X \times \square^1) & \longrightarrow & \mathrm{colim}_{Y \in \mathcal{S}(X|D)} \mathrm{Gr}^p K_0^Y(X). \end{array}$$

Suppose we are given  $\alpha$  as in the statement. According to Corollary 4.6, there exists  $\tilde{\beta} \in F^2 K_0^W(X \times \square^1)/F^{p+1}$  for some  $W \in \mathcal{S}(X|D, 1)$  such that

$$\iota_0^* \tilde{\beta} - \iota_1^* \tilde{\beta} = \alpha \quad \text{in} \quad \mathrm{colim}_{Y \in \mathcal{S}(X|D)} F^2 K_0^Y(X)/F^{p+1}.$$

By Proposition 3.4,  $(\prod_{2 \leq i, j \leq p} w_{|i-j|}) \tilde{\beta}$  lifts to

$$\beta^{(2)} + \beta^{(3)} + \dots + \beta^{(p)} \in \bigoplus_{i=2}^p (F^2 K_0^W(X \times \square^1)/F^{p+1})^{(i)}.$$

By Proposition 3.4 again, we see that

$$\left( \prod_{2 \leq i, j \leq p} w_{|i-j|} \right) (\iota_0^* \beta^{(p)} - \iota_1^* \beta^{(p)}) = \left( \prod_{2 \leq i, j \leq p} w_{|i-j|} \right)^2 \alpha.$$

Since  $(\prod_{i=2}^{p-1} w_{p-i}) \beta^{(p)}$  is in  $\mathrm{Gr}^p K_0^W(X \times \square^1)$ , we are done.  $\square$

*Proof of Theorem 0.1.* Let  $X$  be a regular scheme,  $D$  an effective Cartier divisor on  $X$  and  $p$  an integer. Let us consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{colim}_{Y \in \mathcal{S}(X|D, 1)} \mathrm{Gr}^p K_0^Y(X \times \square^1) & \longrightarrow & \mathrm{colim}_{Y \in \mathcal{S}(X|D)} \mathrm{Gr}^p K_0^Y(X) & \longrightarrow & \mathrm{Gr}^p K_0(X, D) & \longrightarrow & 0 \\ \mathrm{cyc}_1 \uparrow & & \mathrm{cyc}_0 \uparrow & & \mathrm{cyc}_{\mathrm{rel}} \uparrow & & \\ \mathrm{colim}_{Y \in \mathcal{S}(X|D, 1)} \mathrm{CH}_Y^p(X \times \square^1) & \longrightarrow & \mathrm{colim}_{Y \in \mathcal{S}(X|D)} \mathrm{CH}_Y^p(X) & \longrightarrow & \mathrm{CH}^p(X|D) & \longrightarrow & 0. \end{array}$$

Since the bottom row is exact (Proposition 1.16) and the composite of the first two morphisms in the upper row is zero (Theorem 2.2), a morphism  $\mathrm{cyc}_{\mathrm{rel}}$  is induced and it is surjective.

Suppose we are given  $\alpha \in \text{CH}^p(X|D)$  such that  $\text{cyc}_{\text{rel}}(\alpha) = 0$ . By Lemma 4.9 and by a simple diagram chase, there exists  $\beta \in \ker(\text{cyc}_0)$  which lifts

$$\left( \prod_{i=2}^{p-1} w_{p-i} \right) \left( \prod_{2 \leq i, j \leq p} w_{|i-j|} \right)^2 \alpha.$$

By Theorem 4.2,  $(\prod_{i=1}^{p-2} w_i) \beta = 0$ . This proves that  $\alpha$  is killed by  $(\prod_{i=1}^{p-2} w_i)^2 (\prod_{2 \leq i, j \leq p} w_{|i-j|})^2$ , whose prime factors are at most  $(p-1)$  by Lemma 3.2.

For the last statement (comparison between  $F^*$  and the gamma filtration), note that  $F^2 K_0(X, D) = F_\gamma^2 K_0(X, D)$  by Theorem 4.7 and then apply Proposition 3.4 to  $F^2 = F_\gamma^2$ .  $\square$

## 5. THE CASE OF $K_1$ -REGULAR MODULUS

Recall that a scheme  $X$  is called  $K_1$ -regular if  $K_1(X) \simeq K_1(X \times \mathbb{A}^n)$  for all  $n \geq 1$ . For example, reduced affine normal crossing schemes are  $K_1$ -regular. Note that affine  $K_1$ -regular implies reduced.

**Chow groups with topological modulus.** Here, we show that if  $D$  is  $K_1$ -regular then the Chow group with modulus  $\text{CH}^*(X|D)$  becomes much simpler (at least up to torsion). Compare the following definition with Notation 1.12 and Definition 1.14.

**Definition 5.1.** Let  $X$  be a well-codimensional catenary scheme,  $D$  an effective Cartier divisor and  $p$  an integer.

- (1)  $\mathcal{S}(X|D_{\text{top}}) := \mathcal{S}(X|D)$  and  $\mathcal{Z}^p(X|D_{\text{top}}) := \mathcal{Z}^p(X|D)$ .
- (2)  $\mathcal{S}(X|D_{\text{top}}, 1)$  is the set of all closed subsets of  $X \times \square^1$  not meeting  $D \times \square^1$ .
- (3)  $\mathcal{Z}^p(X|D_{\text{top}}, 1)$  is the free abelian group with generators  $[W]$ , one for each  $w \in (X \times \square^1 \setminus \{0, 1\})^{(p)}$  whose closure  $W$  in  $X \times \square^1$  does not meet  $D \times \square^1$ .
- (4) We define

$$\text{CH}^p(X|D_{\text{top}}) := \text{coker}(\mathcal{Z}^p(X|D_{\text{top}}, 1) \xrightarrow{\iota_0^* - \iota_1^*} \mathcal{Z}^p(X|D_{\text{top}})).$$

*Remark 5.2.* The groups  $\text{CH}^p(X|D_{\text{top}})$  and its higher variant have been studied in [Mi19, IKr $\infty$ ] by the name of *naïve Chow groups with modulus* and *Chow groups with topological modulus* respectively.

**Lemma 5.3.** *Let  $X$  be a well-codimensional catenary scheme,  $D$  an effective Cartier divisor on  $X$  and  $p$  an integer. Then the sequence*

$$\text{colim}_{Y \in \mathcal{S}(X|D_{\text{top}}, 1)} \mathcal{Z}_Y^p(X \times \square^1) \xrightarrow{\iota_0^* - \iota_1^*} \text{colim}_{Y \in \mathcal{S}(X|D_{\text{top}})} \text{CH}_Y^p(X) \xrightarrow{\epsilon} \text{CH}^p(X|D_{\text{top}}) \longrightarrow 0$$

*is exact.*

*Proof.* Same as Proposition 1.16.  $\square$

The following is a variant of Theorem 2.2, which is a special case of a more general result in [IKr $\infty$ ].

**Lemma 5.4.** *Let  $X$  be a scheme and  $D$  an effective Cartier divisor on  $X$  admitting an affine open neighborhood in  $X$ . Assume that  $X$  is  $K_0$ -regular and that  $D$  is  $K_1$ -regular. Then the sequence*

$$\text{colim}_{Y \in \mathcal{S}(X|D_{\text{top}}, 1)} K_0^Y(X \times \square^1) \xrightarrow{\iota_0^* - \iota_1^*} \text{colim}_{Y \in \mathcal{S}(X|D_{\text{top}})} K_0^Y(X) \xrightarrow{\epsilon} K_0(X, D) \longrightarrow 0$$

*is exact.*

*Proof.* By the assumption,  $K_0(X, D) \simeq K_0(X \times \square^1, D \times \square^1)$ , from which it follows that the composite  $\epsilon \circ (\iota_0^* - \iota_1^*)$  is zero. The surjectivity of  $\epsilon$  follows from Lemma 2.3, and the exactness at the middle term follows from Lemma 2.7.  $\square$

**Theorem 5.5.** *Let  $X$  be a regular scheme and  $D$  an effective Cartier divisor on  $X$ . Assume that  $D$  is  $K_1$ -regular and admits an affine open neighborhood in  $X$ . Then, for each integer  $p$ , there exists a surjective group morphism*

$$\mathrm{CH}^p(X|D_{\mathrm{top}}) \rightarrow F^p K_0(X, D)/F^{p+1} K_0(X, D)$$

whose kernel is  $(p-1)!^N$ -torsion for some positive integer  $N$  depending only on  $p$ .

*Proof.* This follows from Lemma 5.3 and Lemma 5.4 as in §4.  $\square$

**Theorem 5.6.** *Let  $X$  be a regular scheme and  $D$  an effective Cartier divisor on  $X$ . Assume that  $D$  is  $K_1$ -regular and admits an affine open neighborhood in  $X$ . Then, for each integer  $p$ , the canonical surjective morphism*

$$\mathrm{CH}^p(X|D) \rightarrow \mathrm{CH}^p(X|D_{\mathrm{top}})$$

is a  $(p-1)!^N$ -isomorphism for some positive integer  $N$  depending only on  $p$ .

*Proof.* This follows from Theorem 0.1 and Theorem 5.5.  $\square$

**On higher  $K$ -groups.** Let  $A$  be a regular ring. We consider the pair  $(\Delta_A^n, \partial\Delta_A^n)$  of the  $n$ -th simplex over  $A$  and the union of its faces. Under the well-known isomorphism

$$K_0(\Delta_A^n, \partial\Delta_A^n) \simeq K_n(A),$$

the relative coniveau filtration on  $K_0(\Delta_A^n, \partial\Delta_A^n)$  induces a filtration  $F^*$  on  $K_n(A)$ . In [B194], Bloch studied this filtration and showed that it coincides with the  $\gamma$ -filtration on  $K_n(A)$  rationally. Theorem 0.1 (or Theorem 5.5) gives an interpretation of its graded pieces by Chow groups with (topological) modulus.

**Theorem 5.7.** *Let  $A$  be a regular ring of dimension  $d$  and  $n, p$  non-negative integers. Then there exists a surjective group morphism*

$$\mathrm{CH}^p(\Delta_A^n|\partial\Delta_A^n) \twoheadrightarrow F^p K_n(A)/F^{p+1} K_n(A)$$

whose kernel is  $(p-1)!^N$ -torsion for some positive integer  $N$  depending only on  $p$ . The same is true if we replace  $\mathrm{CH}^p(\Delta_A^n|\partial\Delta_A^n)$  by  $\mathrm{CH}^p(\Delta_A^n|(\partial\Delta_A^n)_{\mathrm{top}})$ . Furthermore, the filtration  $F^*$  coincides with the  $\gamma$ -filtration on  $K_n(A)$  up to  $(d+n-1)!^M$ -torsion for some positive integer  $M$  depending only on  $d+n$ .

*Proof.* We apply Theorem 0.1 and Theorem 5.5 to the pair  $(\Delta_A^n, \partial\Delta_A^n)$ ; note that  $\partial\Delta_A^n$  is  $K_1$ -regular. Then the result follows from the aforementioned isomorphism  $K_0(\Delta_A^n, \partial\Delta_A^n) \simeq K_n(A)$ . This is proved, for example, by a sequence of isomorphisms

$$K_0(\Delta_A^n, \partial\Delta_A^n) \simeq KH_0(\Delta_A^n, \partial\Delta_A^n) \simeq KH_n(A) \simeq K_n(A),$$

where  $KH$  denotes the homotopy invariant  $K$ -theory. The first isomorphism follows from the  $K_1$ -regularity of  $\Delta_A^n$  and  $\partial\Delta_A^n$ , the second one follows from the excision property of  $KH$ , and the last one follows from the homotopy invariance of  $K(A)$ .  $\square$

*Remark 5.8.*  $\mathrm{CH}^*(\Delta_A^n|\partial\Delta_A^n)$  or  $\mathrm{CH}^*(\Delta_A^n|(\partial\Delta_A^n)_{\mathrm{top}})$  is not quite the same as Bloch's higher Chow group  $\mathrm{CH}^*(\mathrm{Spec} A, n)$  in [B186]. However, if  $A$  is a smooth algebra over a field, then by Theorem 5.7 and [loc. cit., Theorem 9.1] we get rational isomorphisms

$$\mathrm{CH}^*(\Delta_A^n|\partial\Delta_A^n)_{\mathbb{Q}} \simeq \mathrm{CH}^*(\Delta_A^n|(\partial\Delta_A^n)_{\mathrm{top}})_{\mathbb{Q}} \simeq \mathrm{CH}^*(\mathrm{Spec} A, n)_{\mathbb{Q}}.$$

We do not know whether the comparison holds integrally or not.

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